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S.S. Moskaliuk
Bogolyubov Institute for Theoretical Physics
Metrolohichna Street, 14-b, Kyiv-143, Ukraine, UA-03143

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University of Neyshabur
P.O. Box 91136-899, Neyshabur, Iran
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Department of Pure Mathematics
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Orla 171, 30-244 Krakow, Poland
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and
Atronomical Observatory
Odessa National University
St. Dvoryanskaya 2, 65082 Odessa, Ukraine
Alexander Zhuk,
Atronomical Observatory
Odessa National University
St. Dvoryanskaya 2, 65082 Odessa, Ukraine
Laszlo Jenkovszky
BITP, Academy of Science of Ukraine
03680 Kiev, Ukraine
and
Wigner Research Centre for Physics
Budapest, Hungary
jenk@bitp.kiev.ua

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INTRODUCTION TO THE BASIC CATEGORY THEORY*

S.S. Moskaliuk
Bogolyubov Institute for Theoretical Physics
Metrolohichna Street, 14-b, Kyiv-143
Ukraine, UA-03143
mss@bitp.kiev.ua

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Abstract

The great revolution of the 20th century started with the theory of special and general relativity and culminated in quantum theory. However, up to date, there are still some fundamental issues with quantum theory that are, yet, to be solved. A possible way to overcome critical issues in present- day quantum physics is through a reformulation of astroparticle physics and quantum theory in terms of a different mathematical framework called category theory.

Parts 1 and 2 introduce the basic notions of category theory, functors and natural transformations. Examples of Representations of categories are presented in Part 3. In Part 4 there is introduced the very important concept of Multiplicative Structures on Categories. In Part 5 there are formulated the basic definitions of the Method of Additional Structures on Objects of a Category, on the one hand, and Topological Quantum Field Theory, on the other hand.

This is one of a text to address all of basic aspects of category theory at the graduate student level.

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1 Categories, monoids and groupoids

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [1–7]. These morphisms are required to satisfy certain properties which make the set of all such relations coherent. Given a category, it is not the case that every two objects have a relation between them, some do and others dont. For the ones that do, the number of relations can vary depending on which category we are considering.

DEFINITION 1.1. *A category is a quadruple* $(\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *consisting of:*

(C1) a class Ob *of objects;*

(C2) for each ordered pair (A, B) *of objects a set* $\mathrm{Hom}(A, B)$ *of morphisms;*

(C3) for each object A *a morphism* $\mathrm{id}_A \in \mathrm{Hom}(A, A)$*, the identity of* A*;*

(C4) a composition law associating to each pair of morphisms $f \in \mathrm{Hom}(A, B)$ *and* $g \in \mathrm{Hom}(B, C)$ *a morphism* $g \circ f \in \mathrm{Hom}(A, C)$*;*

which is such that:

(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathrm{Hom}(A, B)$*,* $g \in \mathrm{Hom}(B, C)$ *and* $h \in \mathrm{Hom}(C, D)$*;*

(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in Hom(A, B)$*;*

(M3) the sets $\mathrm{Hom}(A, B)$ *are pairwise disjoint.*

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B, however it can always be satisfied by replacing $\mathrm{Hom}(A, B)$ by the set $\mathrm{Hom}(A, B) \times (\{A\}, \{B\})$.

EXAMPLE 1.1. *The classic example is* **Set**, *the category with sets as objects and functions as morphisms, and the usual composition of functions as composition.*

To understand this definition, we should ask first: what is a category with one object? It is a — *monoid.* We shall also give the definition of a

monoid in [7] in more details like this: a set M with an associative binary product and a unit element 1 such that $a1 = 1a = a$ for all a in M. Monoids abound in mathematics; they are in a sense the most primitive interesting algebraic structures.

To check that a category with one object is "essentially just a monoid", note that if our category $\mathbf{C}$ has one object x, the set $\mathrm{Hom}(x, x)$ of all morphisms from x to x is indeed a set with an associative binary product, namely composition, and a unit element, namely id_x.

How about categories in which every morphism is invertible? We say a morphism $f : x \to y$ in a category has inverse $g : y \to x$ if $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Well, a category in which every morphism is invertible is called a *groupoid.*

Finally, a group is a category with one object in which every morphism is invertible. It's both a monoid and a groupoid!

When we use groups in physics to describe symmetry, we think of each element g of the group G as a "process". The element 1 corresponds to the "process of doing nothing at all". We can compose processes g and h — do h and then g — and get the product $g \circ h$. Crucially, every process g can be "undone" using its inverse g^{-1}.

So: a monoid is like a group, but the "symmetries" no longer need be invertible; a category is like a monoid, but the "symmetries" no longer need to be composable.

A morphism $a\colon A \to B$ is called *isomorphism* if there exists a morphism $b\colon B \to A$ such that $a \circ b = i_B$ and $b \circ a = i_A$. In this case objects A and B are called *isomorphic.*

Morphisms $a\colon D \to A$ and $b\colon D \to B$ are called *isomorphic* if there exists an isomorphism $c\colon A \to B$ such that $c \circ a = b$.

An object Z is called *terminal* object if for any object A there exists a unique morphism from A to Z, which is denoted $z_A\colon A \to Z$ in what follows.

It is said that a category has (pairwise) products if for every pair of objects A and B there exists their *product*, that is, an object $A \times B$ and a pair of morphisms $\pi_{A,B}\colon A \times B \to A$ and $\nu_{A,B}\colon A \times B \to B$, called projections,

such that for any object D and for any pair of morphisms $a\colon D \to A$ and $b\colon D \to B$ there exists a unique morphism $c\colon D \to A \times B$, satisfying the following conditions:

$$\pi_{A,B} \circ c = a, \qquad \nu_{A,B} \circ c = b. \tag{1}$$

We call such morphism c the *product of morphisms* a and b and denote it $a * b$.

It is easily seen that existence of products in a category implies the following equality:

$$(a * b) \circ d = (a \circ d) * (b \circ d). \tag{2}$$

In a category with products, for two arbitrary morphisms $a\colon A \to C$ and $b\colon B \to D$ one can define the morphism $a \times b$:

$$a \times b\colon A \times B \to C \times D, \qquad a \times b \stackrel{\text{def}}{=} (a \circ \pi_{A,B}) * (b \circ \nu_{A,B}). \tag{3}$$

This definition and (1) obviously imply that the morphism $c = a \times b$ satisfy the following conditions:

$$\pi_{C,D} \circ c = a \circ \pi_{A,B}, \qquad \nu_{C,D} \circ c = b \circ \nu_{A,B}. \tag{4}$$

Moreover, $c = a \times b$ is the only morphism satisfying conditions (4).

It is also easily seen that (2) and (3) imply the following equality:

$$(a \times b) \circ (c * d) = (a \circ c) * (b \circ d). \tag{5}$$

Suppose $A \times B$ and $B \times A$ are two products of objects A and B taken in different order. By the properties of products, the objects $A \times B$ and $B \times A$ are isomorphic and the natural isomorphism is

$$\sigma_{A,B}\colon A \times B \to B \times A, \qquad \sigma_{A,B} \stackrel{\text{def}}{=} \nu_{A,B} * \pi_{A,B}. \tag{6}$$

Moreover, for any object D and for any morphisms $a\colon D \to A$ and $b\colon D \to B$, the morphisms $a * b$ and $b * a$ are isomorphic, that is,

$$\sigma_{A,B} \circ (a * b) = b * a. \tag{7}$$

Similarly, by the properties of products, the objects $(A \times B) \times C$ and $A \times (B \times C)$ are isomorphic. Let

$$\alpha_{A,B,C}\colon (A \times B) \times C \to A \times (B \times C)$$

be the corresponding natural isomorphism. Its "explicit" form is:

$$\alpha_{A,B,C} \stackrel{\text{def}}{=} (\pi_{A,B} \circ \pi_{A\times B,C}) * \Big((\nu_{A,B} \circ \pi_{A\times B,C}) * \nu_{A\times B,C}\Big). \tag{8}$$

Then for any object D and for any morphisms $a\colon D \to A$, $b\colon D \to B$, and $c\colon D \to C$ we have

$$\alpha_{A,B,C} \circ ((a * b) * c) = a * (b * c). \tag{9}$$

REMARK 1.1. *If* $\mathbf{A} = (\mathrm{Ob}, \mathrm{Hom}, \mathrm{id}, \circ)$ *is a category, then*
1) The class Ob *of* **A***-objects is usually denoted by* $\mathrm{Ob}(\mathbf{A})$.
2) The class of all **A***-morphisms (denoted by* $\mathrm{Mor}(\mathbf{A})$*) is defined to be the union of all the sets* $\mathrm{Hom}(A, B)$ *in* **A**.
3) If $A \xrightarrow{f} B$ *is an* **A***-morphism, we call* A *the* **domain** *of* f *[and denote it by* $dom(f)$*] and call* B *the* **codomain** *of* f *[and denote it by* $cod(f)$*]. Observe that condition (M3) guarantees that each* **A***-morphism has a* **unique** *domain and a* **unique** *codomain. However, this condition is given for technical convenience only, because whenever all other conditions are satisfied, it is easy to "force" condition (M3) by simply replacing each morphism* f *in* $\mathrm{Hom}(A, B)$ *by a triple* (A, f, B). *For this reason, when verifying that an entity is a category, we will disregard condition (M3).*
4)The composition, $\circ$, *is a partial binary operation on the class* $\mathrm{Mor}(\mathbf{A})$. *For a pair* (f, g) *of morphisms,* $f \circ g$ *is defined if and only if the domain of* f *and the codomain of* g *coincide.*
5) If more than one category is involved, subscripts may be used for clarification (as in $\mathrm{Hom}_{\mathbf{A}}(A, B)$*).*

2 Functors and natural transformations

DEFINITION 2.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **covariant functor** *from a category* $\mathbf{X}$ *to a category* $\mathbf{Y}$ *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\mathcal{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$*, and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$*.*

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.
- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 2.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **contravariant functor** *from a category* $\mathbf{X}$ *to a category* $\mathbf{Y}$ *is a family of functions* $\mathcal{F}$ *which associates to each object* A *in* $\mathbf{X}$ *an object* $\mathcal{F}A$ *in* $\mathbf{Y}$ *and to each morphism* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *a morphism* $\mathcal{F}f \in \mathrm{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$*, and which is such that:*

(FI) $\mathcal{F}(g \circ f) = \mathcal{F}f \circ \mathcal{F}g$ *for all* $f \in \mathrm{Hom}_{\mathbf{X}}(A, B)$ *and* $g \in \mathrm{Hom}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathrm{Ob}(\mathbf{X})$*.*

Thus, a contravariant functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant functor is also called a presheaf.

So far we have defined categories and maps between them called functors. We will now abstract one step more and define maps between functors. These are called *natural transformations.*

DEFINITION 2.3. *Let* $\mathcal{F} : \mathbf{X} \to \mathbf{Y}$ *and* $\mathcal{G} : \mathbf{X} \to \mathbf{Y}$ *be two functors. A* **natural transformation** $\alpha : \mathcal{F} \to \mathcal{G}$ *is given by the following data:*

For every object A in $\mathbf{X}$ there is a morphism $\alpha_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathbf{Y}$ such that for every morphism $f : A \to B$ in $\mathbf{X}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{G}(A) \\ \mathcal{F}(f)\downarrow & & \downarrow\mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{G}(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{G}(f) \circ \alpha_A = \alpha_B \circ \mathcal{F}(f)$.

The morphisms α_A, $A \in \mathrm{Ob}(\mathbf{X})$, are called the *components of the natural transformation* α.

So, we can certainly speak, as before, of the "equality" of categories. We can also speak of the *isomorphism of categories*: an isomorphism between $\mathbf{C}$ and $\mathbf{D}$ is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ for which there is an inverse functor $\mathcal{G} : \mathbf{D} \to \mathbf{C}$. I.e., $\mathcal{F}\mathcal{G}$ is the identity functor on $\mathbf{C}$ and $\mathcal{G}\mathcal{F}$ the identity on $\mathbf{D}$, where we define the composition of functors in the obvious way. But because we also have natural transformations, we can also define a subtler notion, the *equivalence of categories*. An equivalence is a functor $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ together with a functor $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ and natural isomorphisms $a : \mathcal{F}\mathcal{G} \to 1_{\mathbf{C}}$ and $b : \mathcal{G}\mathcal{F} \to 1_{\mathbf{D}}$. A *natural isomorphism* is a natural transformation which has an inverse.

As we can "relax" the notion of equality to the notion of isomorphism when we pass from sets to categories, we can relax the condition that $\mathcal{F}\mathcal{G}$ and $\mathcal{G}\mathcal{F}$ equal identity functors to the condition that they be isomorphic to the identity functor when we pass from categories to the 2-category **Cat**.

We need to have the natural transformations to be able to speak of functors being isomorphic, just as we needed functions to be able to speak of sets being isomorphic. In fact, with each extra level in the theory of n-categories, we will be able to come up with a still more refined notion of "n-equivalence" in this way.

Thus, in contrast to a set, which consists of a static collection of "things", a category consists not only of objects or "things" but also morphisms which can viewed as "processes" transforming one thing into another. Similarly,

in a 2-category, the 2-morphisms can be regarded as "processes between processes", and so on. The eventual goal of basing mathematics upon omega-categories is thus to allow us the freedom to think of any process as the sort of thing higher-level processes can go between. By the way, it should also be very interesting to consider "$\mathbb{Z}$-categories" (where $\mathbb{Z}$ denotes the integers), having j-morphisms not only for $j = 0, 1, 2, ...$ but also for negative j. Then we may also think of any thing as a kind of process.

It is also possible to combine all these definitions in a coherent way and define the category of functors. In particular, given two categories $\mathbf{C}$ and $\mathbf{D}$, the collection of all covariant (or contravariant) functors $F : \mathbf{C} \to \mathbf{D}$ is actually a category which will be denoted as $\mathbf{D}^{\mathbf{C}}$. This is called the *category of functors* or *functor category* and has as objects covariant (or contravariant) functors and as map natural transformations between functors.

3 Representations of categories

As it was shown in §1 to define a category $\mathbf{K}$ we require the following data:

(a) a set $\mathrm{Ob}(\mathbf{K})$ of elements called the objects of the category $\mathbf{K}$;

(b) for any two objects $Z, W \in \mathrm{Ob}(\mathbf{K})$ a set $\mathrm{Mor}_{\mathbf{K}}(Z, W)$ is defined, called the *morphisms* from Z to W (when it is clear what the category in question is, we omit the index $\mathbf{K}$ and merely write $\mathrm{Mor}(Z, W)$);

(c) for any $P \in \mathrm{Mor}(Z, Z')$ and $Q \in \mathrm{Mor}(Z', Z'')$ their product is defined $QP \in \mathrm{Mor}(Z, Z'')$. The product must be associative: the formula

$$R(QP) = (RQ)P$$

holds for any $P \in \mathrm{Mor}(Z, Z')$, $Q \in \mathrm{Mor}(Z', Z'')$, and $R \in \mathrm{Mor}(Z'', Z''')$;

(d) it is usually assumed that the set $\mathrm{Mor}(Z, Z)$ contains an element 1_Z called the *identity* such that, for any $P \in \mathrm{Mor}(Y, Z)$, we have $P \cdot 1_Z = P$ and, for any $Q \in \mathrm{Mor}(Z, W)$, we have $1_Z \cdot Q = Q$.

EXAMPLE 3.1. *The objects of the category are the finite-dimensional complex linear spaces, and the morphisms are the linear operators.*

EXAMPLE 3.2. *The objects of the category are the groups. The morphisms are the group homomorphisms. The categories of rings, algebras, fields, semigroups, and so on, are introduced in a similar fashion.*

EXAMPLE 3.3. *Let G be a fixed group. The objects of the category are the representations of G, and the morphisms are the intertwining operators (see [7]).*

The set $\mathrm{Mor}\,(Z, Z)$ is also denoted by $\mathrm{End}(Z)$ (or $\mathrm{End}_{\mathbf{K}}(Z)$). It is a semigroup and the elements of $\mathrm{End}(Z)$ are called *endomorphisms*. The invertible endomorphisms of an object Z form a group $\mathrm{Aut}_A(Z)$ and its elements are called the *automorphisms* of the object Z. Finally, the set $\mathrm{Mor}_{\mathbf{K}}$ of all morphisms of the category $\mathbf{K}$ forms a so-called *groupoid*. We recall that by a *groupoid* we mean a set with a partially defined operation: if the products ab and $(ab)c$ are defined, then bc and $a(bc)$ are also defined and $(ab)c = a(bc)$.

It is customary to think of categories as belonging to a different level in the mathematical hierarchy from groups, rings, algebras, and so on; see, for example, Shafarevich [8]. (Even from the point of view of formal logic, it is usual to regard the set of objects of a category as a proper class rather than a set.) However, in this book we shall make no distinction, dealing with categories in the same way as with groups, rings, and so on [6, 9, 10].

3.1 Category of linear relations

The objects of this category are linear spaces over a field $\mathbb{F}$ (and we will suppose them to be finite-dimensional). The morphisms $P : Z \rightrightarrows W$ are the *linear relations*, that is, the subspaces P of $Z \oplus W$.

Sometimes such subspaces are the graphs of linear operators from Z into W, but in general this is not the case.

If $P : Z \rightrightarrows W$ and $Q : W \rightrightarrows Y$ are linear relations, then their product $QP : Z \rightrightarrows Y$ is defined as follows: $(z, y) \in Z \oplus Y$ is contained in the subspace QP if there exists $\omega \in W$ such that $(z, \omega) \in P$ and $(\omega, y) \in Q$ (and this is how one would want to define the product of "multivalued maps").

The following are defined for a linear relation $P : Z \rightrightarrows W$ in the same way as for an operator:

(a) the *kernel* $\ker P$ — the set of all $z \in Z$ such that $(z, 0) \in P$;

(b) the *image* $\operatorname{im} P$ — the projection of P onto W;

(c) the *domain of definition* $D(P)$ — the projection of P onto Z.
In addition we define

(d) the *indefiniteness* $\operatorname{Indef}(P)$; this is the set of $\omega \in W$ such that $(0, \omega) \in P$;
if P is the graph of an operator then $\operatorname{Indef}(P) = 0$;

(e) the *rank* $\operatorname{rk}(P)$:
$\operatorname{rk}(P) = \dim D(P) - \dim \ker P = \dim \operatorname{im} P - \dim \operatorname{Indef} P =$
$= \dim P - \dim \ker P - \dim \operatorname{Indef} P.$

3.2 Representations of categories

A *covariant functor* $(\mathcal{F}, \phi)$ from a category $\mathbf{K}$ to a category $\mathbf{L}$ is determined by the following data:

(i) a map $\mathcal{F} : \operatorname{Ob}(\mathbf{K}) \to \operatorname{Ob}(\mathbf{L})$;

(ii) the collection of maps $(\phi_{Z,W} : \operatorname{Mor}(Z, W) \to \operatorname{Mor}(\mathcal{F}(Z), \mathcal{F}(W))$ is defined for all $Z, W \in \operatorname{Ob}(\mathbf{K})$, and these maps must satisfy the condition that

$$\phi_{Z,Y}(PQ) = \phi_{W,Y}(P)\phi_{Z,W}(Q), \qquad \phi(1_Z) = 1_{\mathcal{F}(Z)}.$$

Similarly, by a *contravariant functor* from a category $\mathbf{K}$ to a category $\mathbf{L}$, we mean a map $\mathcal{F} : \operatorname{Ob}(\mathbf{K}) \to \operatorname{Ob}(\mathbf{L})$ and a collection of maps

$$\phi_{Z,W} : \operatorname{Mor}(Z, W) \to \operatorname{Mor}(\mathcal{F}(W), \mathcal{F}(Z)),$$

defined for all $Z, W \in \mathrm{Ob}(\mathbf{K})$, such that

$$\phi_{Z,Y}(PQ) = \phi_{Z,W}(Q)\phi_{W,Y}(P).$$

EXAMPLE 3.4. *Let* $\mathbf{A}$ *be the category of linear spaces and operators (see Example 3.1), and let* $\mathbf{K}$ *be the category of associative algebras and homomorphisms. We construct a functor* $\mathcal{F} : \mathbf{A} \to \mathbf{K}$. *We associate with each* $Z \in \mathrm{Ob}(\mathbf{A})$ *the exterior algebra* $\mathcal{F}(Z)$ *and with each operator* $L : Z \to W$ *the corresponding natural map of exterior algebras.*

By a *representation of a category* $\mathbf{K}$ we mean a covariant functor $(\mathcal{T}, \tau)$ from $\mathbf{K}$ to the category $\mathbf{A}$. In other words, we associate with each $Z \in \mathrm{Ob}(\mathbf{K})$ a linear space $\mathcal{T}(Z)$, and with each morphism $P : Z \to W$ an operator $\tau(P)$ from $\mathcal{T}(Z)$ to $\mathcal{T}(W)$, such that for each triple $(Z, W, Y) \in \mathrm{Ob}(\mathbf{K})$, and for each $P \in \mathrm{Mor}(Z, W)$ and $Q \in \mathrm{Mor}(W, Y)$, we have

$$\tau(QP) = \tau(Q)\tau(P).$$

Contravariant functions from $\mathbf{K}$ to Op are called *antirepresentations.*

EXAMPLE 3.5. *Let* $\mathbf{X}$ *be the category of finite sets and maps. We define* $\mathcal{T}(X)$ *as the space of functions on* X *and we define* $\tau(p)f(x) = f(p(x))$ *for each map* $p : X \to Y$. *Then* $(\mathcal{T}, \tau)$ *is an antirepresentation.*

Any mathematician, if he delves into his memory, can recall many examples of representations of categories. As a reminder we give three examples (although they are not of great significance as far as this book is concerned).

EXAMPLE 3.6. *Let* $\mathbf{K}$ *be the category of smooth* n*-dimensional manifolds and smooth maps. Let* $\mathcal{T}_k(M)$ *be the space of differential forms of degree* k *on* M, *and let* $\tau_k(P)$ *be the natural maps of differential forms.*

EXAMPLE 3.7. *Let* $\mathbf{K}$ *be the same category as above, and let* $\mathcal{T}_k(M)$ *be the space of* k*-th (de Rham or any other) cohomologies of the manifold, and let* $\tau_k(P)$ *be the natural map of cohomologies.*

EXAMPLE 3.8. *We define the category* $\mathbf{K}$ *as follows. The set* $\mathrm{Ob}(\mathbf{K})$ *consists of two elements* Z, W, *and* $\mathrm{End}(Z)$ *and* $\mathrm{End}(W)$ *consist of the identity element. The set* $\mathrm{Mor}(Z, W)$ *consists of two elements, and* $\mathrm{Mor}(W, Z)$ *is empty. The problem of the classification of representations of* $\mathbf{K}$ *is precisely Kronecker's problem on the reduction of a pair of operators from* Z *to* W *to canonical form.*

3.3 Projective representations of categories

By a *projective representation of a category* $\mathbf{K}$ we mean the following. Associated with each $Z \in \mathrm{Ob}(\mathbf{K})$ is a linear space $\mathcal{T}(Z)$, and with each $P \in \mathrm{Mor}(Z, W)$ a linear operator $\tau(P) : \mathcal{T}(Z) \to \mathcal{T}(W)$ such that, for any $P \in \mathrm{Mor}(Z, W)$ and $Q \in \mathrm{Mor}(W, Y)$, we have

$$\tau(QP) = \lambda(Q, P)\tau(Q)\tau(P),$$

where $\lambda(Q, P) \in \mathbb{C}^*$. (We emphasize that $\lambda(Q, P)$ is not equal to 0; this is important.)

In [7] there are many examples of projective representations of categories. Meanwhile we make the following obvious remark concerning the category $\mathbf{A}^*$, defined as follows.

The *category* $\mathbf{A}^*$ has the linear spaces as its objects, and the linear operators defined to within multiplication by a non-zero constant as its morphisms. In essence, a *projective representation of a category* $\mathbf{K}$ is the same as a functor from $\mathbf{K}$ to $\mathbf{A}^*$ [6, 9].

4 Multiplicative Structures on Categories

4.1 Multiplicative categories

The prototype of a category is the category **Sets** of sets and functions. The prototype of a 2-category is the category **Cat** of small categories and functors. **Cat** has more structure than a simple category because we have natural

transformations between functors. This can be viewed in the following way: The extra structure implies that every morphism set $\mathrm{Hom}(C, D)$ in **Cat** is actually not only a set but a category itself where composition and identities in **Cat** are compatible with this categorical structure on the Hom-sets (i.e. composition and identities are functorial with respect to the structure on the Hom-sets). A general category with this kind of extra structure is called a *2-category*.

The definition of a 2-category can be put in a more general setting (which will be convenient below) by using the language of enriched categories. A category **C** is enriched over a category **V** if every Hom-set in **C** has the structure of an object in **V** and if composition and identities in **C** are compatible with this extra structure on the Hom-sets. So, a 2-category is a category enriched over **Cat**. Now, the (small) 2-categories again form a category **2-Cat** and a *3-category* can be defined as a category enriched over **2-Cat** (indeed, **2-Cat** turns out to be a 3-category itself). In this way we can proceed iteratively to define *n-categories* and then *ω-categories* as categories involving n-categorical structures of all levels.

A concrete recipe to obtain monoidal (braided etc.) 2-categories via *Hopf categories* has been proposed by Crane and Frenkel [11]. Namely, it is supposed that the 2-category of module-categories over a Hopf category now plays an important role in 4-dimensional topology and topological quantum field theory (TQFT) [12]. Although the theory of Hopf categories is formulated, in general, by Neuchl [13], interesting examples are still missing. In particular the Hopf category, underlying Lusztig's canonical basis [14] of a quantized universal enveloping algebra, is not constructed yet. We propose to define it as a family of abelian categories of perverse l-adic sheaves equipped with some functors of multiplication and comultiplication. These perverse sheaves are equivariant in the sense of Bernstein and Lunts [15].

It turns out that the notions of n-category and ω-category are not general enough for several interesting applications. What one gets there are weak versions of these concepts (instead of weak n-category sometimes the notions *bicategory*, *tricategory*, etc. are used). Let us shortly explain what this

means: In a category it does not make sense to ask for equality of objects but the appropriate notion is isomorphism. In the same way, in a 2-category we should not ask for equality of morphisms but only for equality up to an invertible 2-morphism (the morphisms between the morphisms, e. g. the *natural transformations* in **Cat**). Applying this to the categorical structure itself (i.e. requiring associativity and identity properties only up to natural equivalence) leads to the notion of *weak 2-category* (or *bicategory*). In the same way, we can weaken the structure of an n-category up to the $(n-1)$-th level to obtain a *weak n-category.*

The point making this weakening an involved matter is that in general we need so called coherence conditions in addition to the weakened laws in order to assure that some properties, known from the strict case, hold. E.g., to assure that associativity is iteratively applicable (i.e. that we can up to a 2-isomorphism rebracket composites involving more than three factors), we need a coherence condition stating that even four factors can be rebracketed (and the other cases follow then). See the literature [11–15,18] for the details.

A satisfactory version of a weak n-category for higher n and of a *weak ω-category* was not available for a long time but now there are several approaches at hand [16,17]. The relationship between these approaches and a universal understanding of these structures has still to be achieved [1,2].

DEFINITION 4.1. *A* **multiplication** *in the category* $\mathbf{C}$ *is an associative functor*

$$* : \mathbf{C} \times \mathbf{C} \to \mathbf{C} : (X, Y) \mapsto X * Y.$$

An **associativity morphism** *for $*$ is a functor isomorphism*

$$\varphi_{X,Y,Z} : X * (Y * Z) \to (X * Y) * Z$$

such that for any four objects X, Y, Z, T the following diagram is commutative:

$$\begin{array}{ccccc} X*(Y*(Z*T)) & \xrightarrow{\varphi_{X,Y,Z*T}} & (X*Y)*(Z*T) & \xrightarrow{\varphi_{X*Y,Z,T}} & X*Y*Z*T \\ {\scriptstyle id_X*\varphi_{Y,Z,T}}\downarrow & & & & \uparrow{\scriptstyle \varphi_{X,Y,Z}*id_T} \\ X*((Y*Z)*T) & & \xrightarrow{\varphi_{X,Y*Z,T}} & & (X*(Y*Z))*T \end{array}$$

An **commutativity morphism** *for* $*$ *is a functor isomorphism*

$$\psi_{X,Y} : X*Y \to Y*X$$

such that for any two objects X, Y *we have*

$$\varphi_{X,Y} \circ \varphi_{Y,X} = id_{X*Y} : X*Y \to X*Y.$$

Morphisms associativity φ *and commutativity* ψ *are compatible if for any three objects* X, Y, Z *the following diagram is commutative:*

$$\begin{array}{ccccc} X*(Y*Z) & \xrightarrow{\varphi_{X,Y,Z}} & (X*Y)*Z & \xrightarrow{\psi_{X*Y,Z}} & Z*(X*Y) \\ {\scriptstyle id_X*\psi_{Y,Z}}\downarrow & & & & \uparrow{\scriptstyle \varphi_{Z,X,Y}} \\ X*(Z*Y) & \xrightarrow{\varphi_{X,Z,Y}} & (X*Z)*Y & \xrightarrow{\psi_{X,Z}*id_Y} & (Z*X)*Y \end{array}$$

A pair (U, u) *where* $U \in Obj(\mathbf{C})$ *and an isomorphism* $u : U \to U*U$ *is called a* **unit object** *for* $\mathbf{C}, *$ *if the functor*

$$X \mapsto U*X : \mathbf{C} \to \mathbf{C}$$

is equivalence of categories.

DEFINITION 4.2. *A* **multiplicative category** *is a collection* $(\mathbf{C}, *, \varphi, \psi, U, u)$.

If there are some additional structures on category, then it is usually assumed that product $*$ and others elements of the collection are compatible with these structures.

4.2 $\mathcal{C}$-monoids or multiplicative objects

Let $\mathcal{C} = (\mathbf{C}, *, \varphi, \psi, U, u)$ be a multiplicative category. An *multiplicative object* in $\mathcal{C}$ or *$\mathcal{C}$-monoid* is an object $M \in Ob(\mathbf{C})$ with multiplication $\mu : M * M \to M : (m, m') \mapsto \mu(m, m')$ and an unit $\varepsilon : U \to M$ such that the following axioms are faithful:

(1) Associativity: the following diagram is commutative

$$\begin{array}{ccccc} M*(M*M) & & \xrightarrow{\varphi_{M,M,M}} & & (M*M)*M \\ {\scriptstyle id_M*\mu}\downarrow & & & & \downarrow{\scriptstyle \mu*id_M} \\ M*M & \xrightarrow{\mu} & M & \xleftarrow{\mu} & M*M \end{array}$$

(2) Unit: the following diagram is commutative

$$\begin{array}{ccccc} M & \longrightarrow & U*M & \xrightarrow{\psi_{U*M}} & M*U \\ \| & & {\scriptstyle \varepsilon*id_M}\downarrow & & \downarrow{\scriptstyle id_M*\varepsilon} \\ M & \xrightarrow{\mu} & M*M & = & M*M \end{array}$$

EXAMPLE 4.1. *Let $\mathcal{R}$ be a commutative ring. The category $\mathcal{R}$-**mod** of $\mathcal{R}$ -modules is a multiplicative category under the tensor product $\otimes_{\mathcal{R}}$ with the unit object is the left $\mathcal{R}$-module $\mathcal{R}$. Multiplicative objects in the category is $\mathcal{R}-$algebras with units.*

EXAMPLE 4.2. *A small multiplicative category $\mathcal{C}$ is a multiplicative object of the multiplicative category $Sets//Ob(\mathbf{C})$.*

4.3 Monoidal categories and comonoids

Multiplicative structures may be described in categories as monoids in a monoidal category.

A *monoidal category* $(\mathbf{C}, \otimes, K, \varphi, \ldots)$ consists of:

$\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, $K \in \mathrm{Ob}(\mathbf{C})$ – the unit object,

and the functor-isomorphisms:

$$\varphi_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$\psi_A : A \otimes K \to A, \ldots,$

where $\otimes$ is symmetrical, if there exists a functor-isomorphism

$$\theta_{A,B} : A \otimes B \to B \otimes A.$$

A *monoidal functor* (a *morphism of monoidal categories*) of two monoidal categories is defined by $\mathcal{F} : (\mathbf{C}, \otimes, K) \to (\mathbf{C}', \otimes', K')$ if

$$\mathcal{F}(A \otimes B) \cong \mathcal{F}(A) \otimes' \mathcal{F}(B)$$

and $\mathcal{F}(K) \cong K'$.

EXAMPLE 4.3. *A monoidal category is a monoid in the monoidal category* $(\mathcal{C}\dashv\sqcup, \times)$ *of categories with Cartesian product.*

EXAMPLE 4.4. *A monoidal category is the category* $\mathcal{S}ymm$ *with objects* $[n]$ *for* $n = 0, 1, \ldots$ *and morphisms*

$$\mathcal{S}ymm([n], [m]) = \begin{cases} \emptyset, & \text{if } n \neq m, \\ \Sigma_n, & \text{if } n = m. \end{cases}$$

where Σ_n *is the group of permutations of* $(1, \ldots, n)$. *with the multiplication*

$$* : \mathcal{S}ymm \times \mathcal{S}ymm \to \mathcal{S}ymm$$

such that $[n] * [m] \cong [n+m-1]$ *with the folowing identification of the inputs*

$$(1, \ldots, n) * (\overline{1}, \ldots, \overline{m}) = (1, \ldots, n, \overline{2}, \ldots, \overline{m})$$

which explaines the action of $*$ *on morphisms.*

EXAMPLE 4.5. *Let* $(\mathcal{C}, \otimes, K)$ *and* $(\mathcal{C}', \otimes', K')$ *be two monoidal categories,* $\mathcal{F} \in Ob(\mathcal{C}\mathcal{C}')$ *and* $\mathcal{F}(K) = K'$. *Then for such functors* $\mathcal{F}$ *on the category there is a monoidal structure and a monoid is defined by a functor morphism*

$$\mu_{A,B} : \mathcal{F}(A) \otimes' \mathcal{F}(B) \to \mathcal{F}(A \otimes B)$$

with natural associativity axioms and unit.

5 Method of Additional Structures on Objects of a Category

5.1 Functors as additional structure on categories

5.1.1 A. Forgetful functor

In a category, two objects x and y can be equal or not equal, but they can be *isomorphic* or not, and if they are isomorphic, they can be isomorphic in many different ways. An isomorphism between x and y is simply a morphism $f : x \to y$ which has an inverse $g : y \to x$, such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$.

In the category **Sets** an isomorphism is just a one-to-one and onto function, i.e. a bijection. If we know two sets x and y are isomorphic we know that they are "the same in a way", even if they are not equal. But specifying an isomorphism $f : x \to y$ does more than say x and y are the same in a way; it specifies a *particular way* to regard x and y as the same.

In short, while equality is a yes-or-no matter, a mere *property*, an isomorphism is a *structure*. It is quite typical, as we climb the categorical ladder (here from elements of a set to objects of a category) for properties to be reinterpreted as structures [1–5].

DEFINITION 5.1. *We tell that a functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ define a additional $\mathscr{C}-$structure on objects of the category $\mathbf{C}'$ if*

1. $\forall X, Y \in Ob(\mathbf{C})$ *the map* $F : \mathbf{C}(X, Y) \to \mathbf{C}'(\mathcal{F})X), \mathcal{F}(Y))$ *is injective,*
2. $\forall X \in Ob(\mathbf{C}), Y \in Ob(\mathbf{C}')$ *and an isomorphism* $u : Y \to \mathcal{F}(X)$ *there is an object* $\tilde{Y} \in Ob$ *and an isomorphism* $\tilde{u} : \tilde{Y} \to X$ *such that* $\mathcal{F}(\tilde{Y}) = Y$ *and* $\mathcal{F}(\tilde{u}) = u$.

Such functor is called a **forgetful** *functor.*

Let the category $\mathbf{X}$ be concrete over some category $\mathbf{A}$ in the sense that there exists a faithful functor $\mathcal{U}$ from $\mathbf{X}$ to $\mathbf{A}$, usually called the forgetful

functor. The left adjoint to this functor (if it exists) is then called the free functor. A standard example is the forgetful functor from complete metric spaces to metric spaces, whose left adjoint in the completion functor.

Almost all usual mathematical structures are structures on sets in this sense and there are corresponding forgetful functors to the category **Set** of sets.

A forgetful functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ defines a $\mathscr{C}$-structure *on morphisms of the category* $\mathbf{C}'$.

For our general structures we can define the usual construction:

- inverse and direct images of structures;
- restrictions on subobjects,
- different products of structures.

We can define the category $\mathbf{Str}(\mathscr{C})$ of forgetful functors to the category $\mathbf{C}$. It is a full subcategory of $\mathbf{Cat}/\mathscr{C}$ of all categories over $\mathscr{C}$.

Some properties of structures (= forgetful functors):

- In the category $\mathbf{Str}(\mathscr{C})$ the (bundle) product always exist. It gives a "union" structure.
- Any functor $\mathcal{A} : \mathbf{C} \to \mathbf{C}'$ transfers structures to inverse direction, i.e. it defines the functor

$$\mathcal{H}^* : \mathbf{Str}(\mathscr{C}') \to \mathbf{Str}(\mathscr{C}) : \mathcal{F} \mapsto \mathcal{H}^*\mathcal{F}.$$

- For a forgetful functor $\mathcal{F} : \mathbf{C} \to \mathbf{C}'$ the functors

$$(\mathcal{F}\circ) = Funct(id, \mathcal{F}) : Funct(\mathbf{B}, \mathbf{C}) \to Funct(\mathbf{B}, \mathbf{C}')$$

$$(\circ\mathcal{F}) = Funct(\mathcal{F}, id) : Funct(\mathbf{C}', \mathbf{B}) \to Funct(\mathbf{C}, \mathbf{B})$$

are forgetful functors.

- One of the constructions which transfers the structure $\mathcal{F} : \mathbf{C} \to \mathbf{Set}$ defined on sets to objects of any category $\mathbf{B}$ is the functor

$$\mathcal{G} : \mathbf{B} \to Funct(\mathbf{B}', \mathbf{Set}) : \mathbf{B} \mapsto \mathcal{G}_{\mathbf{B}}.$$

Thus we have

$$\begin{array}{ccc} {\mathcal{G}_{\mathbf{B}}}^*\mathbf{C} & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow \mathcal{F} \\ \mathbf{B}' & \xrightarrow{\mathcal{G}_{\mathbf{B}}} & \mathbf{Set} \end{array}$$

- If a functor $\mathcal{A} : \mathbf{B} \to \mathbf{C}$ is injective on morphisms (the condition (1) of the definition of the forgetful functor) then a forgetful functor $\mathcal{F} : \mathbf{B}' \to \mathbf{C}$ and an equivalence $\mathcal{I} : \mathbf{B} \to \mathbf{B}'$ exist, such that the following diagram is commutative

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\mathcal{A}} & \mathbf{C} \\ \mathcal{I}\downarrow & \nearrow \mathcal{F} & \\ \mathbf{B}' & & \end{array}$$

5.1.2 B. Bundle of categories

Let $\mathcal{E} : \mathbf{F} \to \mathbf{C}$ be a functor and for all objects $U \in Ob(\mathbf{C})$ and we denote by $\mathbf{F}_U = \mathcal{E}^{-1}(U, id_U)$ the subcategory of $\mathbf{F}$ with

$$Ob(\mathbf{F}_U) = \{u \in Ob(\mathbf{F}) |\ \mathcal{E}(u) = U\},$$

$$Mor(\mathbf{F}_U) = \{f \in Mor(\mathbf{F}) |\ \mathcal{E}(f) = id_U\}.$$

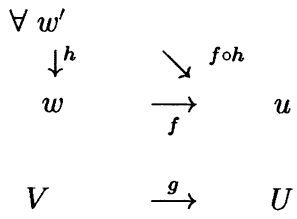
Let $(f : w \to u) \in Mor(\mathbf{F})$, $\mathcal{E}(f : w \to u) = (g : W \to U)$. Then one tells that f is *Descartes's morphism* , or that w is the *inverse image* $g^*(u)$ *of the object* u, if $\forall w' \in Ob(\mathbf{F}_W)$ the map

$$f_* : \mathbf{F}_W(w', w) \to \mathbf{F}_g(w', u) : h \mapsto f \circ h$$

is a bijection. Here we define

$$\mathbf{F}_g(w, u) \stackrel{def}{=} \{h \in \mathbf{F}(w, u) \mid \mathcal{E}(h) = g.\}$$

So we arrive at the diagram



A functor $\mathcal{P} : \mathbf{F} \to \mathbf{C}$ is called a *bundle of categories* if inverse images are allowed to exist and a composition of two Descartes morphism is a Descartes morphism too. Then g^* may be transfered to the functor $\mathcal{G} : \mathbf{F}(U) \to \mathbf{F}(W)$, and $(g_1 \circ g_2)^*$ will be canonical isomorphic to $g_2^* \circ g_1^*$.

EXAMPLE 5.1. *The projection*

$$\Pi_1 : \mathbf{Mor(Top)} \to \mathbf{Top} : (f : X \to Y) \mapsto X$$

is a bundle of categories. For different structures on topological spaces it is not always true for the category of all morphisms, but may be true for a subcategory.

EXAMPLE 5.2. *Let* **Sub** *be a subcategory in* $\mathbf{Mor(Man)}$ *which consists from submersions. Then the projection*

$$\Pi_2 : \mathbf{Sub} \to \mathbf{Man} : (f : X \to Y) \mapsto Y$$

is a bundle of categories and for each morphism $h \in \mathbf{Man}(B', B)$ *we have the functor of the* **inverse image***:*

$$\mathcal{H}^* : \mathbf{Sub}_B \to \mathbf{Sub}_{B'} : (f : M \to B) \mapsto (B' \times_B M \to B').$$

The set $\Gamma(\pi)$ *of sections of an submersion* $\pi : M \to B$ *is the set of morphisms* $\mathbf{Sub}(id_B, \pi)$.

EXAMPLE 5.3. *Let* **Mod** *be the category of pairs* $(\mathcal{R}, M)$ *where* $\mathcal{R}$ *is a ring and* M *is a left* $\mathcal{R}$*-module. Let* **Ring** *be the category of rings. Then the functor*

$$\mathcal{H} : \mathbf{Mod} \to \mathbf{Ring} : (\mathcal{R}, M) \mapsto \mathcal{R}$$

is a bundle of categories and for each morphism $h \in \mathbf{Ring}(\mathcal{R}', \mathcal{R})$ *we have the functor of the* **inverse image***:*

$$\mathcal{H}^* : \mathcal{R}\text{-}\mathbf{mod} \to \mathcal{R}'\text{-}\mathbf{mod} : M \mapsto \mathcal{R}' \otimes_{\mathcal{R}} M.$$

5.1.3 C. Fibers of functor morphisms

Grothendieck's definition of a fiber of a functor morphism is applicable to morphisms of functors from any category to the category **Set** of sets. Let $\mathcal{F}, \mathcal{G} : \mathbf{C} \to \mathbf{Set}$, and $\varphi : \mathcal{F} \to \mathcal{G}$ be their morphism. For each object $S \in Ob(\mathbf{C})$ and an element $\alpha \in \mathcal{G}(S)$ the *fiber* $\mathcal{H}_\alpha$ *of* φ *over* α is the following functor

$$\mathcal{H}_\alpha : \mathbf{C}/S \to \mathbf{Set} : f \mapsto \mathcal{H}_\alpha(f),$$

where for a morphism $f : T \to S$

$$\mathcal{H}_\alpha(f) = \{\beta \in \mathcal{F}(T) \mid \mathcal{G}(f) \circ \mathcal{H}_T(\beta) = \alpha\}.$$

So we have the following diagram

$$\begin{array}{lccc} \mathcal{H}^\alpha(f) \subset & \mathcal{F}(T) & & \mathcal{F}(S) \\ & \mathcal{H}_T \downarrow & & \downarrow \mathcal{H}_S \\ & \mathcal{G}(T) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(S) \quad \ni \alpha. \end{array}$$

5.2 Categorical structures on topological spaces

Among the structures on topological spaces we can select that one, which is compatible with the topology. Let **Top** be a category of some topological spaces with a forgetful functor $\mathcal{F} : \mathbf{Top} \to \mathbf{Set}$.

The categories associated with a topological space $T \in Ob(\mathbf{Top})$ are as follows:

- The category $\mathbf{T}(T)$, where $Ob(\mathbf{T})$ is the set of all open subsets of T, and $Mor(T', T''))$ are all their inclusions.

- The category (pseudogroup) **DE**, where $Ob(\mathbf{DE})$ is the set of all open subsets of T, and $Mor(T', T'')$ are all their homeomorphisms.

Functors $\mathcal{PRESH} : \mathbf{T} \to \mathbf{Set}$ are called *presheaves* of sets on T. Some of them are called sheaves. Thus we have the inclusions

$$\mathcal{SH}(T) \subset \mathcal{PRESH}(T) \subset \mathcal{FUNCT}(\mathbf{T}, \mathbf{Set}).$$

A *Grothendieck topology* on a category is defined by saying which families of maps into an object constitute a *covering* of the object when certain axioms are fulfilled. A category together with a Grothendieck topology on it is called a *site*. For a site $\mathfrak{C}$ one define the full subcategories $\mathcal{SH}(\mathfrak{C}) \subset \mathcal{PRESH}(\mathfrak{C}) \subset \mathcal{FUNCT}(\mathfrak{C}^\circ, \mathbf{Set})$. The objects of $\mathcal{FUNCT}(\mathfrak{C}^\circ, \mathbf{Set})$ are called *presheaves* on the site $\mathfrak{C}$, and the objects of $\mathcal{SH}(\mathfrak{C})$ are called *sheaves* on $\mathfrak{C}$.

For any category there exists the finest topology such that all representable presheaves are sheaves. It is called the *canonical* Grothendieck topology. *Topos* is a category which is equivalent to the category of sheaves for the canonical topology on them.

Hence, the topology is already transfered on a category. So now it is natural to consider on language of toposes and sheaves in all questions connected to local properties.

Here we shall not consider local structures on toposes in general, and we shall restrict ourselves to the consideration of the elementary case of the category **Top**.

DEFINITION 5.2. *A structure defined by a forgetful functor* $\mathcal{F} : \mathbf{C} \to \mathbf{Top}$ *is called a* **local structure** *if*

$\forall C \in Ob(\mathbf{C})$ *and any inclusion map* $i : U \to \mathcal{F}(C)$ *of the open subset* U *an object* $\tilde{U} \in Ob(\mathbf{C})$ *and a morphism* $\tilde{i} \in \mathbf{Mor}(\tilde{U}, C)$ *exist such that* $\mathcal{F}(\tilde{U}) = U$ $\mathcal{F}(\tilde{i}) = i$. *This* $C-$*structure* $\tilde{U}$ *is denoted by* $C|U$ *and called a* **restriction** *of* C *on* U.

In other words we can restrict ourselves to local structures on open subsets.

For a local structure $\mathcal{F} : \mathbf{C} \to \mathbf{Top}$ and each object $X \in Ob(\mathbf{Top})$ there is the presheaf of categories

$$\mathbf{T}(X)^\circ \to \mathbf{Cat} : U \mapsto \mathcal{F}^{-1}(U, id_U).$$

Often this presheaf is a sheaf.

5.3 Categorical structures on smooth manifolds

Let $\mathbf{M}$ be the category of smooth (∞-differentiable) manifolds with forgetful functor $\mathcal{F} : \mathbf{M} \to \mathbf{Top}$, which defines a local structure and the presheaves of these structures are sheaves. On the category $\mathbf{M}$ there is the *tangent* functor $\mathcal{T} : \mathbf{M} \to \mathbf{M} : M \mapsto \mathcal{T}(M)$.

Its iterations give us almost all interesting functors on $\mathbf{M}$. Among them we note the following:

- The cotangent functor $\mathcal{T}^* : \mathbf{M} \to \mathbf{M} : M \mapsto \mathcal{T}^*(M)$.
- For a manifold M and natural number $k = 0, 1, \ldots$ the functor of k–jets $\mathcal{J}^k : \mathbf{M} \to \mathbf{M} : N \mapsto \mathcal{J}^k(M, N)$.
- For a manifold M, $x \in M$, and natural number $k = 0, 1, \ldots$ the functor of k–jets at the point x $\mathcal{J}^k_x : \mathbf{M} \to \mathbf{M} : N \mapsto J^k_x(M, N)$.

Any category $\mathbf{C}$ of structures on smooth manifolds (or on $M/$) has an additional structure, which give us a possibility to define "smooth families of morphisms".

DEFINITION 5.3. *Let $M, M', M'' \in \mathbf{M}$. A map*

$$\Phi : M \to \mathbf{M}(M', M'') : x \mapsto \Phi_x$$

is called a **smooth family** *of morphisms if there exists a smooth map ϕ : $M \times M' \to M''$ such that*

$$\forall x \in M,\ x' \in M' \quad \Phi_x(x') = \phi(x, x').$$

Thus we get the class of categories with smooth families and it appears as the natural condition on functors.

DEFINITION 5.4. *A functor is called a smooth functor if it maps each smooth family to a smooth family.*

Of course all functors $\mathcal{T}, \mathcal{T}^*, \mathcal{J}^k, \mathcal{J}^k_x$ are smooth.

5.4 Double categories as additional structure on categories

5.4.1 A. Definition and examples of double categories

In any category $\mathbf{C}$ with bundle products for some morphisms we can define so-called intern categories. This is a monoid in the multiplicative category $\mathbf{C}/O$ of pairs of (special) morphisms $D, R : M \to O$ with the bundle product:

for $\xi = (D, R : M \to O)$ and $\xi' = (D', R' : M \to O)$ we get $\xi \star \xi' = (D \circ \pi_1, R' \circ \pi_2 : M \times_O M' \to O)$ where the unit objects $\mathrm{id}_M : 0 \to M$ and $\mathrm{id}_{M'} : 0 \to M'$ and the following diagram is commutative

$$\begin{array}{ccc} M \times_O M' & \xrightarrow{\pi_2} & M' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle R'}. \\ M & \xrightarrow{R} & O \end{array}$$

So an intern category is an object $\xi = (D, R : M- > O)$ with a multiplication $\mu : \xi \star \xi' \to \xi$ and the unit $\mathrm{id}_M : O \to M$.

Now we consider such an intern category as the category **Cat** of categories and will call it *double category* [2].

DEFINITION 5.5. *A* **double category D** *consists of the following:*

(1) A category $\mathbf{D}_0$ *of objects* $Ob(\mathbf{D}_0)$ *and morphisms* $Mor(\mathbf{D}_0)$ *of* 0*-level.*

(2) A category $\mathbf{D}_1$ *of morphisms* $Ob(\mathbf{D}_1)$ *of* 1*-level and morphisms* $Mor(\mathbf{D}_1)$ *of* 2*-level.*

(3) Two functors $\mathcal{D}, \mathcal{R} : \mathbf{D}_1 \rightrightarrows \mathbf{D}_0$.

(4) A composition functor

$$* : \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 \to \mathbf{D}_1$$

where the bundle product is defined by commutative diagram

$$\begin{array}{ccc} \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 & \underrightarrow{\pi_2} & \mathbf{D}_1 \\ \pi_1 \downarrow & & \downarrow \mathcal{D} \\ \mathbf{D}_1 & \underrightarrow{\mathcal{R}} & \mathbf{D}_0 \end{array}$$

(5) A unit functor $\mathcal{ID} : D_0 \to D_1$, *which is a section of* $\mathcal{D}, \mathcal{R}$.

The above data are subject to **Associativity Axiom** and **Unit Axiom**. If both of them are fulfilled only up to equivalence, then the double category is called a **weak double category**, but if they are fulfilled strictly, then it is called a **strong double category**.

Now we see that for two objects $A, B \in Ob(\mathbf{D}_0)$ there are 0-level morphisms $\mathbf{D}_0(A, B)$ which we denote by ordinary arrows $f : A \to B$, and 1-level morphisms $D_{(1)}(A, B)$, which we denote by the arrows $\xi : A \Rrightarrow B$ for $A = \mathcal{D}(\xi)$ and $B = \mathcal{R}(\xi)$. So with a 2-level morphism $\alpha : \xi \to \xi'$, where $\xi : A \Rrightarrow B$ and $\xi' : A' \Rrightarrow B'$ we can associate the following diagram

$$\begin{array}{ccc} A & \overset{\xi}{\Rrightarrow} & B \\ \mathcal{D}(\alpha) \downarrow & & \downarrow \mathcal{R}(\alpha) \\ A' & \overset{\xi'}{\Rrightarrow} & B' \end{array} \qquad \longmapsto \qquad \begin{array}{c} \xi \\ \downarrow \alpha \\ \xi' \end{array}$$

and arrow $\alpha : \mathcal{D}(\alpha) \Rrightarrow \mathcal{R}(\alpha)$.

On each level we have the corresponding compositions:

$$\begin{array}{lcl} \text{0-level} & (A \xrightarrow{f} B \xrightarrow{g} C) & \mapsto \quad g \circ f : A \to C \\ & \xi \xrightarrow{\alpha} \eta \xrightarrow{\beta} \varsigma & \mapsto \quad \beta \circ \alpha : \xi \to \varsigma \\ \text{1-level} & (A \overset{\xi}{\Rrightarrow} B \overset{\eta}{\Rrightarrow} C) & \mapsto \quad \eta * \xi : A \Rrightarrow C \\ \text{2-level} & (f \overset{\alpha}{\Rrightarrow} g \overset{\beta}{\Rrightarrow} h) & \mapsto \quad \beta * \alpha : f \Rrightarrow h \end{array}$$

The composition on 2-level is associated with the diagram

$$\begin{array}{ccc} A & \overset{\xi}{\Rightarrow} & B \\ \mathcal{D}(\alpha)\downarrow & & \downarrow\mathcal{R}(\alpha) \\ A' & \overset{\xi'}{\Rightarrow} & B' \\ \mathcal{D}(\alpha')\downarrow & & \downarrow\mathcal{R}(\alpha') \\ A'' & \overset{\xi''}{\Rightarrow} & B'' \end{array} \qquad \longmapsto \qquad \begin{array}{c} \xi \\ \downarrow\alpha \\ \xi' \\ \downarrow\alpha' \\ \xi'' \end{array}$$

Thus, a double category $\mathbf{D}$ consists of

- four sets $Ob(\mathbf{D}_0), Mor(\mathbf{D}_0), Ob(\mathbf{D}_1), Mor(\mathbf{D}_1)$, and eight maps of type $\mathcal{D}, \mathcal{R}$

$$\begin{array}{ccc} Ob(\mathbf{D}_1) & \leftleftarrows & Mor(\mathbf{D}_1) \\ \downarrow\downarrow & & \downarrow\downarrow \\ Ob(\mathbf{D}_0) & \leftleftarrows & Mor(\mathbf{D}_0) \end{array}$$

- two associated categories $\mathbf{D}_0$, $\mathbf{D}_1$, and **almost categories**: $\mathbf{D}_{(2)}$ with the set of objects $Ob(\mathbf{D}_0)$ and the set of morphisms $Ob(\mathbf{D}_1)$, $\mathbf{D}_{(3)}$ with the set of objects $Mor(\mathbf{D}_0)$ and the set of morphisms $Mor(\mathbf{D}_1)$,

- $\mathcal{R}, \mathcal{D} : \mathbf{D}_{(3)} \to \mathbf{D}_{(2)}$ are almost functors.

Now we can define for double categories **double (category) functors** and their **morphisms**, **double subcategories** , the category **DCat** of double categories, **equivalence** of double categories, **dual double categories** (changed direction of 1-level morphisms, i.e. d, r are transposed), and so on.

DEFINITION 5.6. *A* **double category functor** $\mathcal{F} : \mathbf{D} \to \mathbf{D}'$ *is a pair* $\mathcal{F}_0 : \mathbf{D}_0 \to \mathbf{D}'_0, \mathcal{F}_1 : \mathbf{D}_1 \to \mathbf{D}'_1$ *of usual functors such that*

$$\begin{array}{r} \mathcal{D}' \circ \mathcal{F}_1 = \mathcal{F}_0 \circ \mathcal{D}, \quad \mathcal{R}' \circ \mathcal{F}_1 = \mathcal{F}_0 \circ \mathcal{R}, \\ \forall\, \xi, \xi' \in Ob(\mathbf{D}_1) \qquad \varphi_{\xi,\xi'} : \mathcal{F}_1(\xi * \xi') \widetilde{\to} \mathcal{F}_1(\xi) *' \mathcal{F}_1(\xi'), \\ \forall\, A \in Ob(\mathbf{D}_0) \qquad \varphi_A : \mathcal{F}_1(\mathcal{ID}_A) \widetilde{\to} \mathcal{ID}_{\mathcal{F}_0(A)}. \end{array}$$

EXAMPLE 5.4. **Bicategories** *are the partial case of double category* $\mathbf{D}$ *when the category* $\mathbf{D}_0$ *is trivial, i.e. has only identical morphisms and the composition of 1-level and 2-level morphisms are associative.*

EXAMPLE 5.5. *For each category* $\mathbf{C}$ *we have the canonical double category* $\mathbf{Mor}(\mathbf{C})$ *of morphisms. Let* $\mathbf{C}$ *be a category,* T *be the diagram* $\bullet \to \bullet$, $T\mathbf{C}$ *be the category of diagrams in* $\mathbf{C}$ *of type* T, *and let* $\mathbf{D}_0 = C$ *and* $\mathbf{D}_1 = TC$. *The functor* $\mathcal{D}$ *maps the diagram* $f : A \to B$ *into the object* A, *the functor* $\mathcal{R}$ *maps this diagram into the object* B, *and so on. It is easy to see that we get a double category* $\mathbf{D}$ *which is noted by* $\mathbf{Mor}(\mathbf{C})$. *Here* $Ob(\mathbf{D}_1) = Mor(\mathbf{D}_0)$, *where a 2-level morphism* $f \Rrightarrow g$ *is a pair* (u, v) *of morphisms* $u, v \in Mor(\mathbf{C})$ *from the commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ f\downarrow & & \downarrow f' \\ B & \xrightarrow{v} & B' \end{array}$$

with usual composition

EXAMPLE 5.6. *Let* $\mathbf{C}$ *be a category with bundle products, i.e. for all morphisms* $u, v \to Y$, *and the universal square*

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow & & \downarrow v \\ X & \xrightarrow{u} & Z \end{array}$$

exists. And let T *be the following diagram*

$$\bullet \leftarrow \bullet \rightarrow \bullet.$$

Then $T\mathbf{C}$ *be the category of diagrams in* $\mathbf{C}$ *of type* T. *Now we define the double category* $\mathbf{D}$ *with* $\mathbf{D}_0 = \mathbf{D}$ *and* $\mathbf{D}_1 = T\mathbf{C}$. *Two functors*

$$\mathcal{D}, \mathcal{R} : T\mathbf{C} \to \mathbf{C},$$

where the functor $\mathcal{D}$ *maps the diagram* $A \leftarrow M \to B$ *into the object* A, *the functor* $\mathcal{R}$ *maps this diagram into the object* B. *For two* 1*-level morphisms*

$\xi = (A \overset{\pi}{\leftarrow} M \overset{f}{\rightarrow} B) : A \Rrightarrow B$ *and* $\xi' = (B \overset{\pi'}{\leftarrow} M' \overset{f'}{\rightarrow} C) : B \Rrightarrow C$ *we define their composition* $\xi' \circ \xi = (A \overset{\pi\circ\pi_1}{\leftarrow} M \times_B M' \overset{f\circ\pi_2}{\rightarrow} C)$ *where the bundle product is defined by the universal diagram*

$$\begin{array}{ccc} M \times_B M' & \overset{\pi_2}{\rightarrow} & M' \\ \pi_1 \downarrow & & \downarrow \pi' \\ M & \overset{f}{\rightarrow} & B \end{array}$$

A 2-level morphism is a triple $\alpha = (u, v, w) : \xi \to \xi'$ *from the following commutative diagram*

$$\begin{array}{ccccc} M & \overset{f}{\rightarrow} & B & & \\ \pi \downarrow & \searrow v & & \searrow w & \\ A & & M' & \overset{f'}{\rightarrow} & B' \\ & \searrow u & \pi' \downarrow & & \\ & & A' & & \end{array}$$

with the evident composition.

EXAMPLE 5.7. *Let us consider a multiplicative (tensor) category* $(\mathbf{C}, \otimes, U, u)$ *(see §4.1). Then we have the double category with* $\mathbf{D}_1 = \mathbf{C}$, *and* $\mathbf{D}_0 = (*, *)$, *i.e., a trivial category with one object and one morphism. The composition is*

$$\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 = \mathbf{C} \times \mathbf{C} \overset{\otimes}{\rightarrow} \mathbf{C}.$$

Let us consider it in more details. Let $(\mathbf{C}, \otimes, U, u)$ *be a multiplicative (tensor) category with multiplication*

$$\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C} : (X, Y) \mapsto X \otimes Y,$$

for the functor isomorphism of associativity

$$\varphi : \otimes \circ (id, \otimes) \to \otimes \circ (\otimes, id)$$

we write

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

so the pentagon is commutative

$$\begin{array}{ccccc}
 & & X \otimes (Y \otimes (V \otimes W)) & & \xrightarrow{\varphi_{X,Y,V\otimes W}} \\
 & & id_X \otimes \varphi_{Y,V,W} \downarrow & & \\
 & & X \otimes ((Y \otimes V) \otimes W) & & \\
\xrightarrow{\varphi_{X,Y,V\otimes W}} & (X \otimes Y) \otimes (V \otimes W) & & \xrightarrow{\varphi_{X\otimes Y,V,W}} & ((X \otimes Y) \otimes V) \otimes W \\
 & & \xrightarrow{\varphi_{X,Y,V\otimes W}} & & \varphi_{X,Y,V} \otimes id_W \uparrow \\
 & & & & (X \otimes (Y \otimes V)) \otimes W
\end{array}$$

Then we have the double category $\mathbf{D}$ *with* $\mathbf{D}_0 = \mathbf{C}$ *and* $\mathbf{D}_1$ *such that*

$$Ob(\mathbf{D}_1) = \{(X, x) | A, B, X \in Ob(\mathbf{C}), \quad x : X \otimes A \to B\}.$$

So, we write $\xi = (X, x) : A \Rrightarrow B$ *and for* $\xi \in Ob(\mathbf{D}_1)$ *we denote* $\xi = (X_\xi, x_\xi)$, $\mathcal{D}(\xi) = A_\xi$, $\mathcal{R}(\xi) = B_\xi$. *2-level morphisms*

$$\mathbf{D}_1(\xi, \xi') = \{(f_1, f_2, f_3) \mid \textit{commutative diagram} \begin{array}{ccc} X \otimes A & \xrightarrow{x} & B \\ f_3 \otimes f_1 \downarrow & & \downarrow f_2' \\ X' \otimes A' & \xrightarrow{x'} & B' \end{array} \}$$

and $\mathcal{D}(f_1, f_2, f_3) = f_1$, $\mathcal{R}(f_1, f_2, f_3) = f_2$.

Composition $\mathbf{D}_1 \times_{\mathbf{D}_0} D_1 \to \mathbf{D}_1$ *is defined as follows:*

for $A \overset{\xi}{\Rrightarrow} B \overset{\xi'}{\Rrightarrow} B'$ $\xi \circ \xi' = (A, B', X'X, x'')$, *where* x'' *is the following composition*

$$(X' \otimes X) \otimes A \xrightarrow{\varphi^{-1}_{X',X,A}} X' \otimes (X \otimes A) \xrightarrow{id_{X'} \otimes x} X' \otimes B \xrightarrow{x'} B'.$$

Associativity: For $A \overset{\xi}{\Rrightarrow} B \overset{\xi'}{\Rrightarrow} B' \overset{\xi''}{\Rrightarrow} B''$ *the left column gives* $x_{\xi'' \circ (\xi' \circ \xi)}$, *the right column gives* $x_{(\xi'' \circ \xi') \circ \xi}$

$$
\begin{array}{ccc}
(X'' \otimes (X' \otimes X)) \otimes A & & ((X'' \otimes X') \otimes X) \otimes A \\
\varphi^{-1}_{X'',X'\otimes X,A} \downarrow & & \varphi^{-1}_{X''\otimes X',X,A} \downarrow \\
X'' \otimes ((X' \otimes X) \otimes A) & & (X'' \otimes X') \otimes (X \otimes A) \\
id_{X''} \otimes \varphi^{-1}_{X',X,A} \downarrow & & id_{X''\otimes X'} \otimes x \downarrow \\
X'' \otimes (X' \otimes (X \otimes A)) & & (X'' \otimes X') \otimes B \\
id_{X''} \otimes (id_{X'} \otimes x) \downarrow & & \varphi^{-1}_{X'',X',B} \downarrow \\
X'' \otimes (X' \otimes B) & = & X'' \otimes (X' \otimes B) \\
id_{X''} \otimes x' \downarrow & & id_{X''} \otimes x' \downarrow \\
X'' \otimes B'' & = & X'' \otimes B' \\
x'' \downarrow & & x'' \downarrow \\
B'' & & B''
\end{array}
$$

So we have isomorphism
$(\varphi_{X'',X',X}, id_{A'}, id_{B'}) : \xi'' \circ (\xi' \circ \xi) \to (\xi'' \circ \xi') \circ \xi.$

5.4.2 B. Action of a double category

Double categories are categorical variants of usual monoids (and groups), and thus we have the corresponding variant for their actions. Below the definition of action of a double category $\mathcal{D}, \mathcal{R} : \mathbf{D}_1 \to \mathbf{D}_0$ on categories over $\mathbf{D}_0$ is given. Thus we get an analog of group-theoretical methods in categorical frames.

DEFINITION 5.7. *(Left)* **action of a double category** $\mathcal{D}, \mathcal{R} : \mathbf{D}_1 \to \mathbf{D}_0$ *on a category* $\mathcal{P} : \mathbf{M} \to \mathbf{D}_0$ *over* $\mathbf{D}_0$ *is a functor* φ *such that*

(1) The diagram

$$\begin{array}{ccc} \mathbf{D}_1 \times_{\mathbf{D}_0} M & \stackrel{\varphi}{\to} & \mathbf{M} \\ \mathcal{R} \circ \pi_1 \ \searrow & & \downarrow \mathcal{P} \\ & & \mathbf{D}_0 \end{array}$$

is commutative, where the bundle product $\mathbf{D}_1 \times_{\mathbf{D}_0} M$ *is defined by the diagram*

$$\begin{array}{ccc} \mathbf{D}_1 \times_{\mathbf{D}_0} M & \stackrel{\pi_2}{\to} & \mathbf{M} \\ \pi_1 \downarrow & & \downarrow \mathcal{P} \\ \mathbf{D}_1 & \stackrel{\mathcal{R}}{\to} & \mathbf{D}_0 \end{array}$$

(2) The diagram

$$\begin{array}{ccccc} (\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1) \times_{\mathbf{D}_0} \mathbf{M} & \stackrel{\cong}{\to} & \mathbf{D}_1 \times_{\mathbf{D}_0} (\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{M}) & \stackrel{id_{\mathbf{D}_1} \times_{\mathbf{D}_0} \varphi}{\longrightarrow} & \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{M} \\ \otimes \times_{\mathbf{D}_0} id_{\mathbf{M}} \downarrow & & & & \downarrow \varphi \\ \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{M} & & \stackrel{\varphi}{\longrightarrow} & & \mathbf{M} \end{array}$$

is commutative to an isomorphism and there exists a functor isomorphism φ *such that*

$$\forall\, \xi, \xi' \in Ob(\mathbf{D}_1),\ m \in Ob(\mathbf{M}_1) \quad \varphi_{\xi,\xi',m} : (\xi * \xi') * m \to \xi * (\xi' * m)$$

(3) For the unit functor we have a functor isomorphism $\chi : \varphi \circ (\mathcal{ID} \times id_{\mathbf{M}}) \widetilde{\longrightarrow} id_{\mathbf{M}}$ *or for objects*

$$\forall\, A \in Ob(\mathbf{D}_0),\ m \in Ob(\mathbf{M}_1) \quad \chi_{A,m} : \mathcal{ID}_A * m \widetilde{\longrightarrow} m$$

So we have the map of pair of objects $\xi \in Ob(\mathbf{D}_1)$, $m \in Ob(\mathbf{M})$ $(A \stackrel{\xi}{\Rightarrow} \mathcal{P}(m), m) \mapsto \varphi(\xi, m)$ such that $\mathcal{P}(\varphi(\xi, m)) = A$, and of morphisms $\alpha \in \mathbf{D}_1(\xi, \xi'), u \in \mathbf{M}(m, m')$

$$\begin{array}{ccccc} \xi & A & \stackrel{\xi}{\Rightarrow} & \mathcal{P}(m) & \varphi(\xi, m) \\ \alpha \downarrow & \mathcal{F} = \mathcal{D}(\alpha) \downarrow & & \downarrow \mathcal{R}(\alpha) = \mathcal{P}(u) & \longmapsto \quad \downarrow \varphi(\alpha, u) \\ \xi' & A' & \stackrel{\xi'}{\Rightarrow} & \mathcal{P}(m') & \varphi(\xi', m') \end{array}$$

and here $\mathcal{P}(\varphi(\alpha, u)) = \mathcal{F}$.

The definition of a **right action** is evident.

EXAMPLE 5.8. *Each double category* $\mathbf{D}$ *acts on itself one from left and from right by the composition* $*$.

EXAMPLE 5.9. *Let* $\mathbf{SC}$ *be a subcategory of morphisms of the category* $\mathbf{C}$ *such that for all* $(\pi : M \to B) \in Ob(\mathbf{SC})$ *and all* $(f : B' \to B) \in Mor(\mathbf{C})$ *there exists the universal square*

$$\begin{array}{rccc} f^*M = & B' \times_B M & \xrightarrow{f'} & M \\ & \pi_1 \downarrow & & \downarrow \pi \\ & B' & \xrightarrow{f} & B \end{array}$$

and $\Pi : \mathbf{SC} \to \mathbf{C}$ *is the projection on base, i.e.* $\Pi : (\pi : M \to B) \longmapsto B$. *Then we have the natural action of* $Mor(\mathbf{C})$ *on* $\mathbf{SC}$

$$Mor(\mathbf{C}) \times_C \mathbf{SC} \longrightarrow \mathbf{SC}$$

which maps the pair $((f : B' \to B), (\pi : M \to B))$ *to* $(\pi_1 : f^*M \to B')$.

EXAMPLE 5.10. *Let* $\mathbf{mod}_k$ *be the category of left modules over* k*-algebras and* $\Pi : \mathbf{mod}_k \to \mathbf{Alg}_k$ *be the natural projection, which an object* (A, M) *maps to* A. *There is natural action of* $\mathbf{Alg}_k$ *on* $\mathbf{mod}_k$

$$(\mathbf{Alg}_k)_1 \times_{(\mathbf{Alg}_k)_0} \mathbf{mod}_k \to \mathbf{mod}_k$$

such that for $\xi = N : A \Rightarrow B$ *and* B*-module* M

$$\xi^*(B, M) = (A, N \otimes_B M).$$

EXAMPLE 5.11. **Characteristic classes**. *Let a double category* $\mathbf{G}$ *act on* $\mathcal{P} : \mathbf{M} \to \mathbf{G}_0$, *and there is a contravariant functor* $\mathcal{H} : \mathbf{G}^\circ \to \mathbf{Mor}(\mathbf{M})$. *A characteristic class of* $m \in Ob(\mathbf{M})$ *is* $c(m) \in \mathcal{H}_0(\mathcal{P}(m))$, *such that for all* $\xi : \mathcal{P}(m') \Rightarrow \mathcal{P}(m)$ *we have*

$$c(\xi^* m) = \mathcal{H}_1(\xi)(c(m)).$$

EXAMPLE 5.12. **Equivariant functors**. *Let* $\mathbf{M}$ *be a category of manifolds (topological or smooth),* $\mathbf{L}$ *be a category of locally trivial bundles over objects of* $\mathbf{M}$. *Then* $\mathbf{Mor}(\mathbf{M})$ *acts on* $\mathbf{L}$. *Let* G *be a topological group,* $\mathbf{M}_G$ *be the category of* G*-manifolds,* $\mathbf{P}$ *is the category of principal bundles with structure group* G *over objects of* $\mathbf{M}$. *The functor*

$$\mathbf{P} \times \mathbf{M}_G \to \mathbf{L}$$

which maps (η, F) *to the fiber bundle* $\eta[F]$ *with fiber* F. *This functor is equivariant relatively of action* $\mathbf{Mor}(\mathbf{M})$ *on* $\mathbf{P}$ *and* $\mathbf{L}$.

EXAMPLE 5.13. *Let* $\mathbf{ISO}(\mathbf{C})$ *be a double subcategory of* $\mathbf{Mor}(\mathbf{C})$ *such that* $(\mathbf{ISO}(\mathbf{C}))_0 = \mathbf{C}$ *and* $(\mathbf{ISO}(\mathbf{C}))_1$ *is the full subcategory of* $(\mathbf{Mor}(\mathbf{C}))_1$ *with*

$$Ob((\mathbf{ISO}(\mathbf{C}))_1) = \{f \in Mor(\mathbf{C}) \quad | \quad f \text{ is an isomorphism } \}.$$

For any forgetful functor $\mathcal{F} : \mathbf{C}' \to \mathbf{C}$ *(see the next subsection) the double category* $\mathbf{ISO}(\mathbf{C})$ *acts on* $\mathbf{C}'$ *from the left*

$$\mathbf{ISO}(\mathbf{C}) \times_{\mathbf{C}} \mathbf{C}' \to \mathbf{C}' : (u : B \to F(C), C) \mapsto u^*C$$

and $w^*(u^*C) \cong (u \circ w)^*C$.

5.4.3 C. Cobordism and double categories

Let $\mathbf{M}_d$ be the category of oriented compact d-dimensional smooth manifolds (with boundary) and piecewise smooth maps (the sense of the condition we do not define more exactly here; this may be such continuous maps $f : M \to Y$ that are smooth on a dense open subset $U_f \subset M$) , let $\mathbf{CM}_d$ be its subcategory of closed (with empty boundary) manifolds and smooth maps, $\mathbf{CM}_d \subset \mathbf{M}_d$.

There are the following functors:

(1) Disjoint union

$$\cup : \mathbf{M}_d \times \mathbf{M}_d \to \mathbf{M}_d : (X, Y) \mapsto X \cup Y.$$

(2) Changing of the orientation of manifolds on opposite

$$(-) : \mathbf{M}_d \to \mathbf{M}_d : X \mapsto -X.$$

(3) Boundary operator

$$\partial : \mathbf{M}_{d+1} \to \mathbf{CM}_d : X \mapsto \partial X.$$

(4) Multiplication on the unit segment $I = [0, 1]$

$$I\times \ : \mathbf{CM}_d \to \mathbf{M}_{d+1} : X \mapsto I \times X.$$

Now we define a double category $\mathbf{C}(d)$ with

(1) $\mathbf{C}(d)_0 = \mathbf{CM}_d$.

(2) The 1-level morphisms $\mathbf{C}(d)_{(1)}(X, X')$ is a set of pairs (Y, f) where Z is oriented compact $(d+1)$-dimensional smooth manifold with the boundary ∂Y and f is an diffeomorphism

$$f : (-X) \cup X' \to \partial Y,$$

where $\cup$ notes the disjoint union of $-X$ and X'. Thus we write $(Y, f) : X \Rightarrow X'$.

(3) The composition of $(Y, f) : X \Rightarrow X'$ and $(Y', f') : X' \Rightarrow X''$ is the morphism

$$(Y \cup_{X'} Y', (f|_X) \cup (f'|_{X'})) : X \Rightarrow X'',$$

where $(Y \cup_{X'} Y')$ denotes the union $(Y \cup Y')$ after identification of each point $f(y) \in f(Y)$ with the point $f'(y) \in f'(Y)$ for all $y \in Y$ and smoothing this topological manifold.

(4) The 1-level identical morphism $\mathcal{ID}_X$ is $(X \times [0; 1], id_{(-X)\cup X})$, because $\partial(X \times [0; 1]) = (-X) \cup X$.

(5) The 2-level morphisms of $\mathbf{C}(d)_1(\xi, \xi')$ from $\xi = (Y, f : X' \cup (-X) \to \partial Y) : X \Rightarrow X'$ to $\xi' = (Y', f' : X'' \cup (-X') \to \partial Y') : X' \Rightarrow X''$ are such triples of smooth maps (f_1, f_2, f_3) that the following diagram is commutative

$$\begin{array}{ccccc} (-X) \cup X' & \xrightarrow{f} & \partial Y & \subset Y \\ \downarrow f_1 \cup f_2 & & & \downarrow f_3 \\ (-X') \cup X'' & \xrightarrow{f'} & \partial Y' & \subset Y' \end{array}$$

It easy to see that functors $\cup$ and $(-)$ may be expanded to double category functors

$$\cup : \mathbf{C}(d) \to \mathbf{C}(d),$$
$$(-) : \mathbf{C}(d) \to \mathbf{C}(d)^{\circ}$$

and $(-)$ is an equivalence of the double categories.

Remark. The appearance of the following two formulas for 1-level morphisms is interesting in algebras and cobordisms

$$f : A \otimes_k B^{\circ} \to End_k(N) \qquad f : (-X) \cup Y \to \partial Z,$$

where we have correspondence between the functors

$$\begin{array}{ccc} (\_)^{\circ} & \longleftrightarrow & -(\_), \\ \otimes_k & \longleftrightarrow & \cup, \\ End_k & \longleftrightarrow & \partial. \end{array}$$

5.5 Topological quantum field theory

Topological quantum field theory (TQFT) is a recent development in the interface between physics and mathematics. The mathematical interest in them comes from the hope that they will disclose new phenomena, or at least offer some efficient organization of previously studied invariants like the Jones invariants of links, or the Donaldson invariants of 4-manifolds.

The physical interest comes from their value as examples of categories derived from the representation category of a Hopf algebra in which extensive calculations are possible (see, for example, [7]).

In this case TQFT is a functor $\mathcal{Z}$ from the category $\mathbf{CM}(d)$ of d-dimensional manifolds to the category $\mathbf{H}$ of (usually Hermitian) finite dimensional vector spaces and some axioms are satisfied (see [7,19]). Actualy, the **functor** $\mathcal{Z}$ is a functor between double categories.

Thus, topological quantum field theory in dimension d is a functor

$$\mathcal{Z} : \mathbf{C}(d) \to \mathbf{Mor}(\mathbf{H}),$$

between double categories such that:

(1) the disjoint union in $\mathbf{C}(d)$ go to the tensor product

$$\cup \quad \mapsto \otimes,$$

where $(\_)^* : \mathbf{H} \to \mathbf{H}^\circ$ is the dualization of vector spaces.

(2) changing of orientation in $\mathbf{C}(d)_0$ go to dualization

$$(-) \mapsto (.)^*$$

Thus, as consequence of double categorical functorial properties, we get

(1) for each compact closed oriented smooth d-dimensional manifold $X \in Ob(\mathbf{C}(d)_0)$ the value of the functor $\mathcal{Z}(X)$ is a finite dimensional vector space over the field $\mathbb{C}$ of the complex numbers (usually with Hermitian metric),

(2) for each $(Y, f) : X \Rightarrow X'$ from $Ob(\mathbf{C}(d)_1)$ the value of the functor $\mathcal{Z}(Y, f)$ is a homomorphism $\mathcal{Z}(X) \to \mathcal{Z}(X')$ of (Hermitian) vector spaces,

and the following well known axioms of topological quantum field theory are satisfied:

A(1) (*involutivity*). $\mathcal{Z}(-X) = \mathcal{Z}(X)^*$, where $-X$ denotes the manifold with opposite orientation, and $*$ denotes the dual vector space.

A(2) (*multiplicativity*). $\mathcal{Z}(X \cup X') = \mathcal{Z}(X) \otimes \mathcal{Z}(X')$, where $\cup$ denotes disconnected union of manifolds.

A(3) (*associativity*). For the composition $(Y'', f'') = (Y, f) * (Y', f')$ of cobordisms have to be

$$\mathcal{Z}(Y'', f'') = \mathcal{Z}(Y', f') \circ \mathcal{Z}(Y, f) \in Hom_{\mathbf{C}}(\mathcal{Z}(X), \mathcal{Z}(X'')).$$

(Usually the identifications

$$\mathcal{Z}(X' - X) \cong \mathcal{Z}(X)^* \otimes \mathcal{Z}(X') \cong Hom_{\mathbf{C}}(\mathcal{Z}(X), \mathcal{Z}(X'))$$

allow us to identify $\mathcal{Z}(Y, f)$ with the element $\mathcal{Z}(Y, f) \in \mathcal{Z}(\partial Y)$.

A(4) For the initial object $\emptyset \in Ob(\mathbf{C}(d)_0)$ $\qquad \mathcal{Z}(\emptyset) = \mathbf{C}$.

A(5) (*trivial homotopy condition*). $\mathcal{Z}(X \times [0, 1]) = id_{\mathcal{Z}(X)}$.

TQFTs are interesting as mathematical structures, but it is hard to see them playing direct roles as models of serious physical theories. The application of TQFT ideas to the theory of modern physics has not been carried very far, as yet (see, for example, [20–23]), but it would be interesting to see what can be achieved in this area).

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ON THE COZERO DIVISOR GRAPH OF THE SEMIGROUP $\mathbb{Z}_n$

M. Afkhami*
Department of Mathematics
University of Neyshabur
P.O. Box 91136-899, Neyshabur, Iran
mojgan.afkhami@yahoo.com

K. Khashyarmanesh, S. Mohammadikhah, N. Vaez Moosavi
Department of Pure Mathematics
International Campus of Ferdowsi
University of Mashhad
P.O. Box 1159-91775, Mashhad, Iran
Khashyar@ipm.ir, S.mohamadi783@gmail.com, nazilamoosavi@gmail.com

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Abstract

In this paper, we investigate the structure of the cozero divisor graph of the semigroup Z_n, denoted by $\Gamma'(\mathbb{Z}_n)$. We obtain some graph theoretical properties of $\Gamma'(\mathbb{Z}_n)$ including its genus, eigenvalues and energy of the graph. Also, we determine the number of spanning trees in $\Gamma'(\mathbb{Z}_n)$.

AMS Classification 13A99, 05C10, 05C25
Keywords. Cozero-divisor graph, Semigroup Z_n, Energy of a graph, Planarity

*Corresponding author

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1 Introduction

The zero divisor graph of a commutative ring was introduced by Beck [10], but this work was mostly concerned with coloring of rings. The zero divisor graph of commutative rings has been studied extensively by Anderson, Frazier, Lauve and Livingston (cf. [7] and [8]). In [12], the study of the zero divisor graphs of semigroups was begun. There are several graphs associated to a semigroup, see for example [14] and [15]. Also, in [1], Afkhami and Khashyarmanesh introduced the cozero divisor graph of a commutative ring R, which is denoted by $\Gamma'(R)$, that examines relationships between principal ideals. Method of trees was invended to elucidate some problems of Lie group representations too. It was developed in 1960-1970-th by Vilenkin-Kuznetsov-Smorodinsky (VKS) [16]. The further development of Vilenkin-Kuznetsov-Smorodinsky method to Vilenkin-Kuznetsov-Moskaliuk-Smorodinsky-Smirnov-(VKMSS)-Trees was given by S.S. Moskaliuk and Yu.F. Smirnov [17 – 23] on the base of conception of complementarity of Lie groups in sense of Moshinsky which exists between groups $O(n)$ and $Sp(2, R)$.

In [3], the present authors defined and studied the cozero divisor graph of a commutative semigroup in a general case. In this paper, we study the cozero divisor graph of the semigroup $\mathbb{Z}_n$ with multiplicative operation, denoted by $\Gamma'(\mathbb{Z}_n)$, for some particular values of n. The cozero divisor graph $\Gamma'(\mathbb{Z}_n)$ is an undirected graph whose vertex set consists of all non-zero and non-unit elements of $\mathbb{Z}_n$, and two distinct vertices a and b are adjacent if and only if $a \notin (b)$ and $b \notin (a)$, where for $c \in \mathbb{Z}_n$, (c) is the ideal $c\mathbb{Z}_n$. In Sections 2 and 3 of this paper, we compute some of the numerical invariants of $\Gamma'(\mathbb{Z}_n)$, namely planarity, eigenvalue, energy, Hamiltonian cycle, Eulerian cycle, dominating set, cut set, independent set and genus. Also, we determine the number of spanning trees in $\Gamma'(\mathbb{Z}_n)$. Moreover, we obtain some results about the complement of $\Gamma'(\mathbb{Z}_n)$ and its line graph.

Now we recall some definitions and notations on graphs. A simple graph is a pair $X = (V, E)$, where $V = V(X)$ and $E = E(X)$ are the sets of vertices and edges of X, respectively. A graph X is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if each

pair of distinct vertices is joined by an edge. For a positive integer n, we use K_n to denote the complete graph with n vertices. Also, we say that X is an *empty graph* if no two vertices of X are adjacent, and X is a *null graph* if it contains no vertices. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets, so that no edges has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{n,m}$. A graph on n vertices such that $n-1$ of the vertices have degree one, all of which are adjacent only to the remaining vertex x, is called a *star graph* with center x. A graph is called *planar* if it can be drawn in the plane without any edge crossing. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [11, p. 153]). The *chromatic number* of a graph X, denoted by $\chi(X)$, is the minimal number of colors which can be assigned to the vertices of X in such a way that every two adjacent vertices have different colors. Moreover, we recall that the union of graphs X_1 and X_2, which is denoted by $X_1 \cup X_2$, where X_1 and X_2 are two vertex-disjoint graphs, is a graph with $V(X_1 \cup X_2) = V(X_1) \cup V(X_2)$ and $E(X_1 \cup X_2) = E(X_1) \cup E(X_2)$. Let X be a graph with $V(X) = \{1, \ldots, n\}$. The *adjacency matrix* of X, denoted by $A(X)$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(X)$ are indexed by $V(X)$. If $i \neq j$, then the (i,j)-entry is 0 for nonadjacent vertices i and j, and (i,j)-entry is 1 for adjacent vertices i and j. The (i,i)-entry of $A(X)$ is 0 for $i = 1, \ldots, n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A(X)$. Recall that the *energy* of X is defined as $\varepsilon(X) = \sum_{i=1}^{n} |\lambda_i|$. A subset $S \subseteq V$ is called an *independent set* if the subgraph of X induced by S has no edges. An *independence number* denoted by $\alpha(X)$, is the cardinality of the largest independent set. A *dominating set* for a graph X is a subset D of V such that every vertex not in D is adjacent to at least one vertex of D. The *dominating number*, denoted by $\lambda(X)$, is the number of vertices in a smallest dominating set of X. A *vertex cutset* in a graph X is a set of vertices whose deletion increases the number of connected components of X. The *vertex connectivity* of a connected graph X is the minimum number of vertices in a vertex cutset, and is denoted by $K_0(X)$. A *Hamiltonian cycle* in a graph X is a cycle that contains every vertex of X. Moreover

X is called *Hamiltonian* if it contains a Hamiltonian cycle. A graph is *Eulerian* if there exists a closed trail containing every edge. A connected graph is Eulerian if and only if all its vertex degrees are even (see [11]). For positive integers a and b, we denote the *greatest common divisor* of a and b, by (a, b). Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, \dots, p_k$ are distinct prime numbers and $\alpha_1, \dots, \alpha_k \geq 1$. Then $\varphi(n) = n(1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_k})$ denotes the *Eulerian function* of n. The *complement* of a graph X, which is denoted by $\overline{X}$, is a graph that vertices are the same as the vertices of X, and we have $\{x, y\} \in E(\overline{X})$ if and only if $\{x, y\} \notin E(X)$. The *line graph* $L(X)$ of a graph X, is the graph with $V(L(X)) = E(X)$ and $\{e_1, e_2\} \in E(L(X))$ if and only if e_1 and e_2 are incident to a common vertex in X. If $\{x, y\} \in E(X)$ then, we denote the corresponding vertex of $L(X)$ by $[x, y]$. A *tree* is an undirected graph in which any two vertices are connected by exactly one simple path. A *spanning tree* of a connected, undirected graph X is a tree that contains all vertices of X. A spanning tree of a connected graph X can also be defined as a maximal set of edges of X that contains no cycles, or as a minimal set of edges that connect all vertices.

2 Structure of the graphs $\Gamma'(\mathbb{Z}_n)$, $\overline{\Gamma'(\mathbb{Z}_n)}$ and $L(\Gamma'(\mathbb{Z}_n))$

In this section, we investigate basic properties of $\Gamma'(\mathbb{Z}_n)$, $\overline{\Gamma'(\mathbb{Z}_n)}$ and $L(\Gamma'(\mathbb{Z}_n))$ in some particular cases. Since $V(\Gamma'(\mathbb{Z}_p)) = \{x \in \mathbb{Z}_p \mid (x, p) \neq 1\}$, one can easily see that $\Gamma'(\mathbb{Z}_p)$ is a null graph.

Theorem 2.1. *If $n = p^t$, where $t \geq 2$, then $\Gamma'(\mathbb{Z}_n)$ is an empty graph.*

Proof. Suppose that x and y are arbitrary distinct vertices in $\Gamma'(\mathbb{Z}_{p^t})$. Then $(x, p^t) \neq 1$ and $(y, p^t) \neq 1$. So we have $(x, p), (y, p) > 1$. Now it is easy to see that $x \in (y)$ or $y \in (x)$, which implies that x is not adjacent to y in $\Gamma'(\mathbb{Z}_{p^t})$. Thus $\Gamma'(\mathbb{Z}_{p^t})$ is an empty graph. □

Corollary 2.2. *If $n = p^t$, where $t \geq 2$, then $L(\Gamma'(\mathbb{Z}_n))$ is a null graph and $\overline{\Gamma'(\mathbb{Z}_n)}$ is complete.*

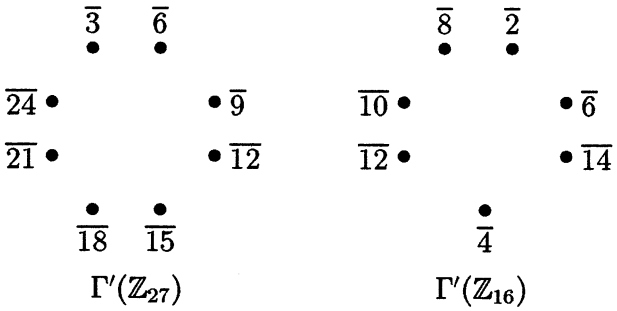


Figure 1

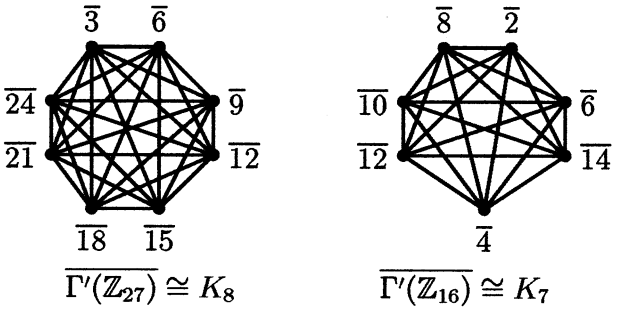


Figure 2

In the following corollary, we study some basic properties of $\Gamma'(\mathbb{Z}_n)$, where $n = p^t$ for $t \geq 2$.

Corollary 2.3. *Let $n = p^t$, where $t \geq 2$. Then the following statements hold.*

(i) $\Gamma'(\mathbb{Z}_n)$ *is planar.*

(ii) If $n \neq 4, 8, 9, 25$, *then* $\overline{\Gamma'(\mathbb{Z}_n)}$ *is not planar.*

(iii) $\chi(\Gamma'(\mathbb{Z}_n)) = 1$ *and* $\alpha(\Gamma'(\mathbb{Z}_n)) = n - \varphi(n) - 1$.

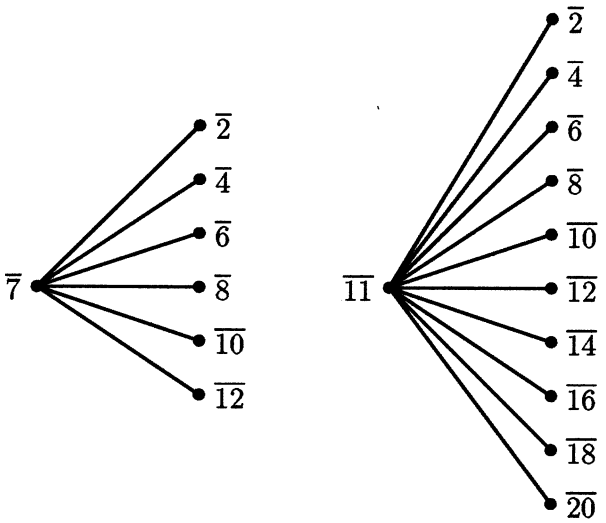
(iv) $\lambda_i = 0$ *for* $1 \leq i \leq n$ *and* $\varepsilon(\Gamma'(\mathbb{Z}_n)) = 0$.

(v) $\Gamma'(\mathbb{Z}_n)$ *is not Hamiltonian and Eulerian.*

In the following theorem, we study the graph $\Gamma'(\mathbb{Z}_n)$, where $n = 2p$ and $p \neq 2$.

Theorem 2.4. *If $n = 2p$, where $p \neq 2$, then $\Gamma'(\mathbb{Z}_n)$ is a star graph.*

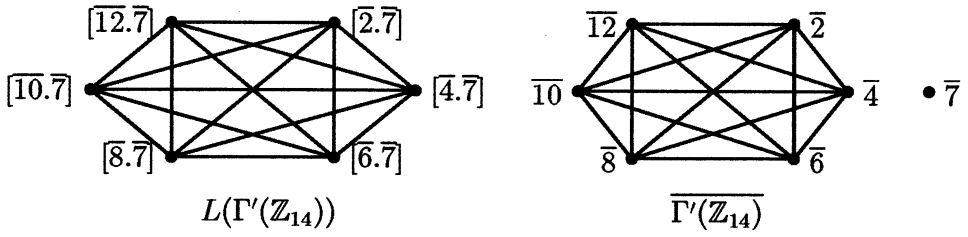
Proof. We claim that the vertex p is adjacent to all vertices y in $\Gamma'(\mathbb{Z}_{2p})$, where $y \neq p$. Since $(y, 2p) \neq 1$, one can easily see that $2 \mid y$. Thus $y = 2z$ for some $z \in \mathbb{Z}_{2p}$. Therefore we have that $p \notin (y)$ and $y \notin (p)$, which implies that p and y are adjacent. Now, let $2k_1$ and $2k_2$ be arbitrary distinct vertices in $\Gamma'(\mathbb{Z}_{2p})$. Clearly $k_1, k_2 < p$. Suppose that $m = k_1 p + k_2 t$, where t is the inverse of k_1 in the field $\mathbb{Z}_p$. Then we have $k_2 \equiv k_1 m \ (mod\ p)$, and so $2k_2 \equiv 2k_1 m \ (mod\ 2p)$. This means that $2k_2 \in (2k_1)$, and so $2k_1$ and $2k_2$ are not adjacent. Hence $\Gamma'(\mathbb{Z}_{2p})$ is a star graph with center p. □



$\Gamma'(\mathbb{Z}_{14}) \cong K_{1,6}$ $\Gamma'(\mathbb{Z}_{22}) \cong K_{1,10}$

Figure 3

Corollary 2.5. *Let $n = 2p$, where $p \neq 2$. Then we have $\Gamma'(\mathbb{Z}_n)) \cong K_{1,p-1}$, $L(\Gamma'(\mathbb{Z}_n)) \cong K_{p-1}$ and $\overline{\Gamma'(\mathbb{Z}_n)} \cong K_1 \cup K_{p-1}$.*



$L(\Gamma'(\mathbb{Z}_{14}))$ $\overline{\Gamma'(\mathbb{Z}_{14})}$

Figure 4

In the following lemma we recall some well-known formulas for eigenvalues and energy of graphs.

Lemma 2.6. *(i) For any positive integer n, the eigenvalues of K_n are $n-1$ with multiplicity 1, and -1 with multiplicity $n-1$.*

(ii) For any positive integers p, q, the eigenvalues of $K_{p,q}$ are $\sqrt{pq}$, $-\sqrt{pq}$ and 0 with multiplicity $p+q-2$.

The following corollary follows easily from Corollary 2.5 and Lemma 2.6.

Corollary 2.7. *Let $n = 2p$, where $p \neq 2$. Then we have the following statements:*

(i) $\Gamma'(\mathbb{Z}_n)$ is planar.

(ii) $L(\Gamma'(\mathbb{Z}_{2p}))$ is planar if and only if $p \in \{3, 5\}$.

(iii) $\chi(\Gamma'(\mathbb{Z}_n)) = 2$, $\alpha(\Gamma'(\mathbb{Z}_n)) = p-1$, $\lambda(\Gamma'(\mathbb{Z}_n)) = 1$ and $K_0(\Gamma'(\mathbb{Z}_n)) = 1$.

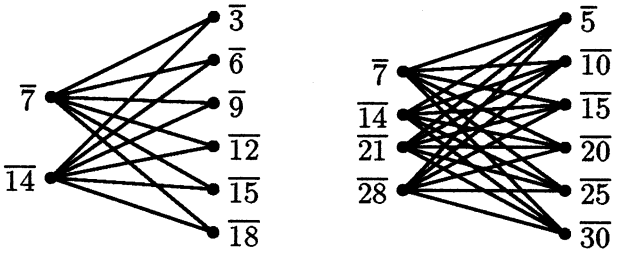
(iv) The eigenvalues of $\Gamma'(\mathbb{Z}_n)$ are $\sqrt{p-1}$, $-\sqrt{p-1}$, and 0 with multiplicity $p-2$, and $\varepsilon(\Gamma'(\mathbb{Z}_n)) = 2\sqrt{p-1}$.

(v) $\Gamma'(\mathbb{Z}_n)$ is not Hamiltonian and Eulerian.

In the following theorem, we investigate the properties of $\Gamma'(\mathbb{Z}_{pq})$, where $p, q > 2$ and $p \neq q$.

Theorem 2.8. *If $n = pq$, where $p, q > 2$ and $p \neq q$, then $\Gamma'(\mathbb{Z}_n) \cong K_{p-1,q-1}$.*

Proof. Put $A = \{x \in \mathbb{Z}_{pq} \ ; \ p \mid x\}$ and $B = \{y \in \mathbb{Z}_{pq} \ ; \ q \mid y\}$. Then clearly $|A| = q-1$ and $|B| = p-1$. Also $A \cup B = V(\Gamma'(\mathbb{Z}_{pq}))$ and $A \cap B = \emptyset$. If $x, x' \in A$, then $x = pt$ and $x' = pt'$, and so clearly $x \in (x')$ or $x' \in (x)$. Hence x is not adjacent to x' in A. Similarly no two distinct vertices in B are adjacent. Now we claim that each vertex in A is adjacency to all vertices in B. Let $x \in A$ be an arbitrary vertex in $\Gamma'(\mathbb{Z}_n)$. We show that x is adjacent to y for every $y \in B$. Since $x \in A$, we have $x = pz$ for some $z \in \mathbb{Z}_q$. Also $y \in B$ implies that $y = qz'$ for some $z' \in \mathbb{Z}_p$. Thus $x \notin (y)$ and $y \notin (x)$. Hence the result holds. □



$\Gamma'(\mathbb{Z}_{21}) \cong K_{2,6}$ $\Gamma'(\mathbb{Z}_{35}) \cong K_{4,6}$

Figure 5

Corollary 2.9. *If $n = pq$, where $p, q > 2$ and $p \neq q$, then $L(\Gamma'(\mathbb{Z}_n))$ contain $p-1$ complete subgraphs isomorphic to K_{q-1}, or $q-1$ complete subgraphs isomorphic to K_{p-1}. Also $\overline{\Gamma'(\mathbb{Z}_n)}$ is isomorphic to $K_{p-1} \cup K_{q-1}$.*

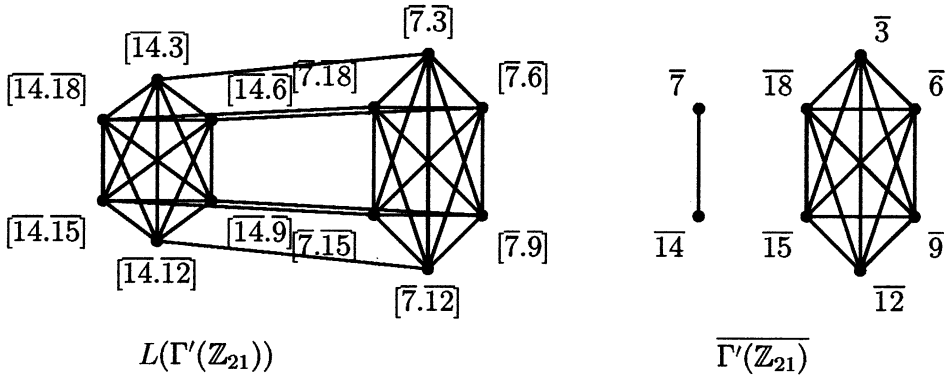
In the following, we study some properties of the graph $\Gamma'(\mathbb{Z}_n)$, where $n = pq$ for $p, q > 2$ and $p \neq q$.

Corollary 2.10. *Let $n = pq$, where $p, q > 2$ and $p \neq q$. Then the following statements hold.*

(i) $\Gamma'(\mathbb{Z}_{pq})$ *is planar if and only if* $p = 3$ *or* $q = 3$.

(ii) $\chi(\Gamma'(\mathbb{Z}_n)) = 2$.

(iii) If $p < q$, *then* $\lambda(\Gamma'(\mathbb{Z}_n)) = p-1$ *and* $\alpha(\Gamma'(\mathbb{Z}_n)) = q-1$.



$L(\Gamma'(\mathbb{Z}_{21}))$ $\overline{\Gamma'(\mathbb{Z}_{21})}$

Figure 6

(iv) The eigenvalues of $\Gamma'(\mathbb{Z}_n)$ *are* $\sqrt{(p-1)(q-1)}$, $-\sqrt{(p-1)(q-1)}$, *and* 0 *with multiplicity* $p+q-4$ *and* $\varepsilon(\Gamma'(\mathbb{Z}_n)) = 2\sqrt{(p-1)(q-1)}$.

(v) $\Gamma'(\mathbb{Z}_n)$ *is not Hamiltonian, but it is Eulerian.*

In the next theorem, we study the graph $\Gamma'(\mathbb{Z}_n)$, whenever $n = p_1 \dots p_k$, where $p_1, \dots, p_k$ are distinct prime numbers and $k \geq 2$.

Theorem 2.11. *Assume that* $p_1, \dots, p_k$, *with* $k \geq 2$, *are distinct prime numbers and* $n = p_1 p_2 \dots p_k$. *Then* $\Gamma'(\mathbb{Z}_n)$ *is* r*-partite, where* $r = \sum_{j=1}^{k-1} \binom{k}{j}$.

Proof. We assume that A_i consists of those vertices x such that the decomposition of x in to prime integers has only one of the $p_i's$, for $1 \leq i \leq k$. Also we suppose that $A_{i_1 \dots i_j}$, where $1 \leq j \leq k-1$, consists of vertices x such that the decomposition of x in to prime integers has exactly j prime numbers $p_{i_1}, \dots, p_{i_j}$, when $p_{i_1}, \dots, p_{i_j} \in \{p_1, \dots, p_k\}$. Now, one can easily see that the number of these parts is equal to $\sum_{j=1}^{k-1} \binom{k}{j}$. Let X_i, where $2 \leq i \leq \sum_{j=1}^{k-1} \binom{k}{j}$ be the parts of $\Gamma'(\mathbb{Z}_n)$. It is enough to show that there is no adjacency in each part. Let X be an arbitrary part such that the decomposition of the elements in X to prime integers, has exactly t prime numbers, and without loss of generality, we denote these

prime numbers by $p_1, \ldots, p_t$, where $1 \leq t \leq k-1$. Let x and y be arbitrary vertices in X. Then $x = p_1 \ldots p_t m$ and $y = p_1 \ldots p_t m_1$, where $m, m_1 \in \mathbb{Z}_{p_{t+1} \ldots p_k}$. We show that x and y are not adjacent. Clearly $(m, p_{t+1}, \ldots, p_k) = (m_1, p_{t+1}, \ldots, p_k) = 1$. Hence m_1 is unit in $\mathbb{Z}_{p_{t+1} \ldots p_k}$. Suppose that r is an integer with $r \equiv m{m_1}^{-1} (mod\ p_{t+1} \ldots p_k)$. So there exists an integer l such that $r = l p_{t+1} \ldots p_k + m{m_1}^{-1}$. Now, we have $ry = ln + p_1 \ldots p_t m$. Thus $x \in (y)$, which implies that x is not adjacent to y. □

Corollary 2.12. *If $n = pqr$, where p, q and r are distinct prime numbers, then $\Gamma'(\mathbb{Z}_n)$ is 6-partite.*

Theorem 2.13. *Suppose that $p_1, \ldots, p_k$ are distinct prime numbers and $n = p_1^t p_2 \ldots p_k$, where $k, t \geq 2$. Then $\Gamma'(\mathbb{Z}_n)$ is r-partite, where $r = \sum_{j=1}^{k} \binom{k}{j}$.*

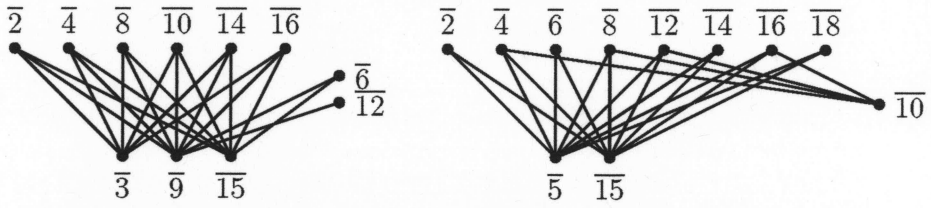
Proof. If $t = 1$, then, by Theorem 2.11, $\Gamma'(\mathbb{Z}_n)$ is $(\sum_{j=1}^{k} \binom{k}{j} - 1)$-partite. Assume that $t \geq 2$. Set $S := \{p_1 p_2 \ldots p_k, p_1^2 p_2 \ldots p_k, \ldots, p_1^{t-1} p_2 \ldots p_k\}$. Let $x = p_1^{\alpha} p_2 \ldots p_k$ and $y = p_1^{\beta} p_2 \ldots p_k$ be two arbitrary vertices in S with $\alpha \geq \beta$. Also let $m = p_1^{\alpha-\beta} - r p_1^{t-\beta}$, where r is a positive integer. Thus we have $x \equiv my\ (mod\ p_1^t p_2 \ldots p_k)$. So $x \in (y)$, which implies that x and y are not adjacent. Therefore $\Gamma'(\mathbb{Z}_n)$ is an r-partite graph whose parts is the union of the parts introduced in Theorem 2.11 and S. Thus the result holds. □

Corollary 2.14. *If $n = p^t q$, where $t \geq 2$, then $\Gamma'(\mathbb{Z}_n)$ is a 3-partite graph. For example, see Figure 7.*

Theorem 2.15. *If $n = p^{t_1} q^{t_2}$, where p and q are distinct prime numbers and $t_1, t_2 \geq 2$, then $\Gamma'(\mathbb{Z}_n)$ is a 4-partite graph.*

Proof. Let $A = \{a \in \mathbb{Z}_n \ ; \ p \mid a\}$, $B = \{b \in \mathbb{Z}_n \ ; \ q \mid b\}$, $C = \{c \in \mathbb{Z}_n \ ; \ pq \mid c\}$, $D = \{d \in \mathbb{Z}_n \ ; \ p^t q \mid d \text{ or } pq^{t'} \mid d\}$, where $2 \leq t \leq t_1$ and $2 \leq t' \leq t_2$. Then clearly $\Gamma'(\mathbb{Z}_n)$ is a 4-partite graph with parts A, B, C and D. □

The following figure shows that $\Gamma'(\mathbb{Z}_{36})$ is a 4-partite graph.



$\Gamma'(\mathbb{Z}_{18})$ $\Gamma'(\mathbb{Z}_{20})$

Figure 7

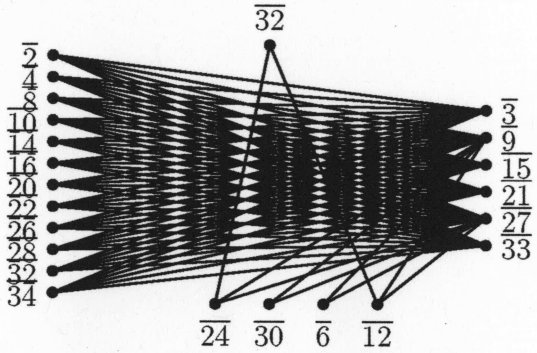


Figure 8

3 The genus and the number of spanning trees in $\Gamma'(\mathbb{Z}_n)$

In this section, first we study the genus of $\Gamma'(\mathbb{Z}_n)$, and then we compute the number of spanning trees in $\Gamma'(\mathbb{Z}_n)$.

We denote S_g for the surface formed by a connected sum of g tori and the number g is called the *genus* of the surface S_g. A simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. The notation $\gamma(X)$ is denoted for the genus of a graph X. Clearly a graph X is planar if $\gamma(X) = 0$, and it is *toroidal* if $\gamma(X) = 1$.

Recall that for a rational number q, $\lceil q \rceil$ is the first integer number greater or equal than q. In the following lemma we bring some well-known formulas

for genus of a graph (see [??, 24]).

Lemma 3.1. *The following statements hold:*

(i) For $n \geq 3$, we have $\gamma(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$.

(ii) For $m, n \geq 2$, we have $\gamma(K_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$.

According to Lemma 3.1, we have $\gamma(K_n) = 0$, for $1 \leq n \leq 4$, and $\gamma(K_n) = 1$, for $5 \leq n \leq 7$, and, for other values of n, $\gamma(K_n) \geq 2$.

In the following lemma, we study the genus of $\Gamma'(\mathbb{Z}_n)$, when $n = p^t$ for $t \geq 2$ and $n = 2p$ with $p \neq 2$.

The following lemma is a immediate consequence of Theorem 2.1 and 2.4

Lemma 3.2. *The following statements hold:*

(i) If $n = p^t$, where $t \geq 2$, then $\gamma(\Gamma'(\mathbb{Z}_n)) = 0$

(ii) If $n = 2p$, where $p \neq 2$, then $\gamma(\Gamma'(\mathbb{Z}_n)) = 0$

The following lemma follows easily from Theorem 2.8 and Lemma 3.1.

Lemma 3.3. *Let $n = pq$, where $p, q > 2$ and $p \neq q$. Then*

$$\gamma(\Gamma'(\mathbb{Z}_n)) = \lceil \frac{1}{4}(p-3)(q-3) \rceil.$$

Now, in the following, we study the genus of $\overline{\Gamma'(\mathbb{Z}_n)}$, where $n = p^t$ for $t \geq 2$.

Lemma 3.4. *Let $n = p^t$, where $t \geq 2$ and $n \neq 4, 9$. Then*

$$\gamma(\overline{\Gamma'(\mathbb{Z}_n)}) = \lceil \frac{1}{12}(n - \varphi(n) - 4)(n - \varphi(n) - 5) \rceil.$$

Proof. By Corollary 2.3, $\overline{\Gamma'(\mathbb{Z}_n)} \cong K_{n-\varphi(n)-1}$, where $n = p^t$ for $t \geq 2$. Clearly, $n - \varphi(n) - 1 \geq 3$, where $n = p^t$ for $t \geq 2$ and $n \neq 4, 9$. Hence the result follows from Lemma 3.1. □

Lemma 3.5. *If $n = 4$ or $n = 9$, then $\gamma(\overline{\Gamma'(\mathbb{Z}_n)}) = 0$.*

Proof. By Corollary 2.2, $\overline{\Gamma'(\mathbb{Z}_4)} \cong K_1$ and $\overline{\Gamma'(\mathbb{Z}_9)} \cong K_2$. Thus one can easily see that $\gamma(\overline{\Gamma'(\mathbb{Z}_n)}) = 0$, where $n = 4$ or $n = 9$. □

Let X be a graph with $V(X) = \{1, 2, \ldots, n\}$. The vertex *Laplacian* matrix of X, denoted by $L(X)$, is the $n \times n$ matrix defined as follows. The rows and columns of $L(X)$ are indexed by $V(X)$. If $i \neq j$, then the (i, j)-entry of $L(X)$ is -1 if i and j are adjacent, and it is 0 if i and j are not adjacent. The (i, i)-entry of $L(X)$ is d_i, the degree of the vertex i, for $i = 1, 2, \ldots, n$. Clearly, $L(X) = D(X) - A(X)$, where $D(X)$ is the diagonal matrix of vertex degrees and $A(X)$ is the adjacency matrix of X. Also, the notation $\kappa(X)$ is denoted for the number of spanning trees of a graph X.

The following lemma, which is from [9], is needed in the rest of the paper.

Lemma 3.6. *Let X be a graph with $V(X) = \{1, 2, \ldots, n\}$. Then the following statements hold:*

(i) The cofactors of any two elements of $L(X)$ are equal.

(ii) The cofactor of any element of $L(X)$ equals the number of spanning trees of X.

(iii) For $n \geq 2$, we have $\kappa(K_n) = n^{n-2}$.

The following lemma follows easily from Theorem 2.1, Corollary 2.3 and Lemma 3.6.

Lemma 3.7. *Let $n = p^t$, where $t \geq 2$. Then*

(i) $\kappa(\Gamma'(\mathbb{Z}_n)) = 0$.

(ii) $\kappa(\overline{\Gamma'(\mathbb{Z}_n)}) = (n - \varphi(n) - 1)^{n-\varphi(n)-3}$

In the following, we study the number of spanning trees of $\Gamma'(\mathbb{Z}_n)$, where $n = 2p$ and $p \neq 2$.

Theorem 3.8. *Let $n = 2p$, where $p \neq 2$. Then*

(i) $\kappa(\Gamma'(\mathbb{Z}_n)) = 1$.

(ii) $\kappa(L(\Gamma'(\mathbb{Z}_n))) = (p-1)^{p-3}$.

Proof. (i) By Corollary 2.5, $\Gamma'(\mathbb{Z}_n) \cong K_{1,p-1}$, where $n = 2p$ and $p \neq 2$. Then

$$L(\Gamma'(\mathbb{Z}_{2p})) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & -1 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ -1 & -1 & \cdots & -1 & p-1 \end{pmatrix}_{p\times p}$$

Hence, by Lemma 3.6, we have

$$\kappa(\Gamma'(\mathbb{Z}_{2p})) = det(L_{\Gamma'(\mathbb{Z}_{2p})}(p|p)) = det(I_{p-1}) = 1$$

(ii) It is follows from Corollary 2.5 and Lemma 3.6.

□

In the following theorem, we investigate the number of spanning trees of $\Gamma'(\mathbb{Z}_{pq})$, where $p = 3$ and $q > 3$.

Theorem 3.9. *Let $n = pq$, where $p = 3$ and $q > 3$. Then*

$$\kappa(\Gamma'(\mathbb{Z}_n)) = (q-1)2^{q-2}.$$

Proof. By Theorem 2.8, $\Gamma'(\mathbb{Z}_{3q}) \cong K_{2,q-1}$, where $q > 3$. Then

$$L(\Gamma'(\mathbb{Z}_{3q})) = \begin{pmatrix} q-1 & 0 & -1 & \cdots & -1 & -1 \\ 0 & q-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \cdots & \vdots \\ -1 & -1 & 0 & \cdots & 2 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 2 \end{pmatrix}_{(q+1)\times(q+1)}$$

Hence, by Lemma 3.6 and elementary row operations, we have

$$\kappa(\Gamma'(\mathbb{Z}_{3q})) = det(L_{\Gamma'(\mathbb{Z}_{3q})}(1|1)) = det \begin{pmatrix} q-1 & -1 & \cdots & -1 & -1 \\ -1 & 2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \vdots \\ -1 & 0 & \cdots & 2 & 0 \\ -1 & 0 & \cdots & 0 & 2 \end{pmatrix}_{q\times q}$$

$$= -det \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 2 \\ -1 & 2 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & 2 & 0 \\ q-1 & -1 & \cdots & 0 & -1 & -1 \end{pmatrix}_{q\times q}$$

Thus we have that

$$\kappa(\Gamma'(\mathbb{Z}_{3q})) = -det \begin{pmatrix} -1 & 0 & \cdots & 0 & 2 \\ 0 & 2 & 0 & \cdots & -2 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 2 & -2 \\ 0 & \cdots & 0 & 0 & q-1 \end{pmatrix}_{q\times q} = (q-1)2^{q-2}.$$

This completes the proof. □

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POLYNORMIAL INVARIANTS OF TORUS KNOTS AND (p,q)-CALCULUS[1]

Anatoliy M. Pavlyuk
Bogolyubov Institute for Theoretical Physics
National Academy of Sciences of Ukraine
Metrolohichna str. 14b, Kyiv 03680, Ukraine
pavlyuk@bitp.kiev.ua

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Abstract

We introduce the deformed fermionic numbers, corresponding to the skein relations, the main characteristics of knots and links. These fermionic numbers allow one to restore the skein relations. For the Alexander (Jones) skein relation we introduce corresponding Alexander (Jones) fermionic q- numbers, and for the HOMFLY skein relation – the HOMFLY deformed (p, q)-numbers with one fermionic parameter.

Keywords: knots and links; Alexander, Jones, HOMFLY skein relations; polynomial invariants; recurrence relation; deformed fermionic numbers.

Mathematics Subject Classification 2010: 57M27, 17B37, 81R50.

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1 Introduction

The problem of searching, investigating and physical interpreting of the skein relations for knots and links is rather difficult one. Because of this we propose to deal with the corresponding deformed (bosonic, fermionic) numbers, instead of working with the skein relations.

In the recent paper [1] we have shown that the one-parameter Alexander (Jones) skein relation is characterized by the corresponding Alexander (Jones) bosonic q-numbers. The two-parameter HOMFLY skein relation is described by the two-parameter HOMFLY bosonic (p, q)-numbers. The introduced bosonic q-numbers/(p, q)-numbers allow one to restore corresponding skein relations.

In the present paper we show that the Alexander (Jones) skein relation can be defined by the corresponding Alexander (Jones) fermionic q-numbers as well. The HOMFLY skein relation is described by the two-parameter HOMFLY deformed (p, q)-numbers with one fermionic parameter. The introduced deformed fermionic numbers allow us to restore the skein relations as well.

2 Skein Relation

Skein relation, the most important characteristics of knots and links, unites the three polynomials $P_{L_+}(q)$, $P_{L_O}(q)$, $P_{L_-}(q)$, and in the general form can be written as

$$P_{L_+}(q) = l_1 P_{L_O}(q) + l_2 P_{L_-}(q), \tag{1}$$

where l_1, l_2 are coefficients. The capital letter "L" stands for "Link", i.e. knot or link. The L_+, L_O, L_- denote Link with overcrossing, Link with zero crossing and Link with undercrossing correspondingly. The initial Link L_+ turns into a simpler Link L_O by the surgery operation of elimination applied to chosen overcrossing of L_+. The same initial Link L_+ turns into another simpler Link L_- by the surgery operation of switching applied to the same chosen overcrossing of L_+.

Applying the surgery operations to $L_{n+1,2}$, the simplest torus knots (if $(n+1)$ is odd) and links (if $(n+1)$ is even integer number), one obtains

$L_{n,2}$ (by elimination) and $L_{n-1,2}$ (by switching). Expressing it in terms of (1), we have the recurrence relation [2]

$$P_{L_{n+1,2}}(q) = l_1 P_{L_{n,2}}(q) + l_2 P_{L_{n-1,2}}(q)\,, \tag{2}$$

which has the same coefficients as the skein relation (1), or in simpler form

$$P_{n+1,2}(q) = l_1 P_{n,2}(q) + l_2 P_{n-1,2}(q)\,. \tag{3}$$

The axiomatic basis of the knot theory includes the skein relation (1), and the normalization condition

$$P_{unknot} = P_{1,2} = 1. \tag{4}$$

3 Two-parameter deformed numbers

The two-parameter (P, Q)-number for an integer n appearing in connection with two-parameter deformed oscillator is defined as [3]

$$[n]_{P,Q} = \frac{P^n - Q^n}{P - Q}\,. \tag{5}$$

Some of the first (P, Q)-numbers are given here:

$[1]_{P,Q}$=1, $[2]_{P,Q}$=P+Q, $[3]_{P,Q}$=P^2+PQ+Q^2, $[4]_{P,Q}$=P^3+P^2Q+PQ^2+Q^3, ...

The recurrence relation for (P, Q)-numbers looks as

$$[n+1]_{P,Q} = (P+Q)[n]_{P,Q} - PQ[n-1]_{P,Q}\,. \tag{6}$$

Comparing (3) (or directly (1)) with (6) allows one to introduce $q-$numbers, corresponding to the skein relation (1). For this it is necessary to find P and Q from

$$P + Q = l_1, \quad PQ = -l_2, \tag{7}$$

and put them into (5).

4 Alexander q-numbers

The Alexander skein relation [4]

$$\Delta_+(q) - \Delta_-(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Delta_O(q) \tag{8}$$

defines the Alexander polynomials $\Delta(q)$ for knots and links. Rewriting it in the form (1)

$$\Delta_+(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Delta_O(q) + \Delta_-(q) \tag{9}$$

gives the "Link coefficients"

$$l_1^A = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad l_2^A = 1\,. \tag{10}$$

In analogy to (3), we have the recurrence relation for Alexander polynomials of torus knots and links $L_{n,2}$:

$$\Delta_{n+1,2}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\Delta_{n,2}(q) + \Delta_{n-1,2}(q)\,. \tag{11}$$

Comparing (11) and (6) one has two equations

$$P + Q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}, \quad PQ = -1\,, \tag{12}$$

from which it follows

$$P = q^{\frac{1}{2}}, \quad Q = -q^{-\frac{1}{2}}\,. \tag{13}$$

Putting (13) into (5), we obtain the Alexander deformed fermionic $q-$numbers

$$[n]^A_{q^{\frac{1}{2}},-q^{-\frac{1}{2}}} = \frac{(q^{\frac{1}{2}})^n - (-q^{-\frac{1}{2}})^n}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}\,. \tag{14}$$

The Alexander $q-$numbers $[n]^A_{q^{\frac{1}{2}},-q^{-\frac{1}{2}}}$, which are described by the below indices, satisfy the following recurrence relation

$$[n+1]^A_{q^{\frac{1}{2}},-q^{-\frac{1}{2}}} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})[n]^A_{q^{\frac{1}{2}},-q^{-\frac{1}{2}}} + [n-1]^A_{q^{\frac{1}{2}},-q^{-\frac{1}{2}}},$$

from which we find the Alexander "Link coefficients" (10), and by putting them into (3) and (1) we obtain (11) and (9).

The Alexander polynomial invariants for torus knots $T(n,l)$ are given by the known formula [5]

$$\Delta_{n,l}(q) = \frac{(q^{\frac{nl}{2}} - q^{-\frac{nl}{2}})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(q^{\frac{l}{2}} - q^{-\frac{l}{2}})}, \tag{15}$$

where n and l are coprime positive integers. For $l = 2$, Eq. (15) gives for torus knots $T(n,2)$:

$$\Delta_{n,2}(q) = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}, \quad n = 2m-1, \tag{16}$$

where m are positive integers: $m = 1, 2, 3, \ldots$. Thus, from (16) and (14) for torus knots $T(n,2)$:

$$\Delta_{n,2}(q) = [n]^A_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}, \quad n = 2m-1. \tag{17}$$

It is easy to verify that (14) in the case of even n gives the formula of the Alexander polynomials for torus links $L(n,2)$:

$$\Delta_{n,2}(q) = [n]^A_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}} = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}, \quad n = 2m. \tag{18}$$

Thus, uniting (16) and (18), one has the Alexander polynomials for torus knots and links $L_{n,2}$:

$$\Delta_{n,2}(q) = [n]^A_{q^{\frac{1}{2}}, -q^{-\frac{1}{2}}}, \quad n = 1, 2, 3 \ldots . \tag{19}$$

So, we introduced the Alexander fermionic $q-$numbers (14), allowing to find the Alexander skein relation (9). At last, we recall that the Alexander bosonic $q-$numbers, which also lead to the Alexander skein relation, were introduced as well [1]:

$$[n]^A_{q,q^{-1}} = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{20}$$

They coincide with $q-$numbers of Biedenharn and Macfarlane [6, 7]. From the recurrence relation for the Alexander bosonic q-numbers (20)

$$[n+1]^A_{q,q^{-1}} = (q + q^{-1})[n]^A_{q,q^{-1}} - [n-1]^A_{q,q^{-1}}$$

one has the "knot coefficients":

$$k_1^A = q + q^{-1},\ k_2^A = -1\,,$$

from which the "Link coefficients" (10), defining the Alexander skein relation (9), can be found with the help of the formulas

$$l_2^A = +(-k_2^A)^{\frac{1}{2}},\quad l_1^A = +(k_1^A - 2l_2^A)^{\frac{1}{2}}.$$

5 Jones q-numbers

The Jones skein relation [8]

$$q^{-1}V_+(q) - qV_-(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_O(q) \tag{21}$$

introduces the Jones polynomials $V(q)$ for knots and links. From (21) in the form (1)

$$V_+(q) = q(q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_O(q) + q^2V_-(q) \tag{22}$$

one finds the Jones "Link coefficients"

$$l_1^V = q^{\frac{3}{2}} - q^{\frac{1}{2}},\quad l_2^V = q^2\,. \tag{23}$$

Comparing (22) and (6) gives

$$P + Q = q^{\frac{3}{2}} - q^{\frac{1}{2}},\quad PQ = -q^2\,, \tag{24}$$

from which

$$P = q^{\frac{3}{2}},\quad Q = -q^{\frac{1}{2}}\,. \tag{25}$$

Putting (25) into (5), we obtain the Jones fermionic $q-$numbers

$$[n]^V_{q^{\frac{3}{2}},-q^{\frac{1}{2}}} = \frac{(q^{\frac{3}{2}})^n - (-q^{\frac{1}{2}})^n}{q^{\frac{3}{2}} + q^{\frac{1}{2}}}, \tag{26}$$

which satisfy the following recurrence relation

$$[n+1]^V_{q^{\frac{3}{2}},-q^{\frac{1}{2}}} = (q^{\frac{3}{2}} - q^{\frac{1}{2}})[n]^V_{q^{\frac{3}{2}},-q^{\frac{1}{2}}} + q^2[n-1]^V_{q^{\frac{3}{2}},-q^{\frac{1}{2}}},$$

defining the Jones "Link coefficients" (23), and, therefore, the Jones skein relation (22) follows from (1).

Finally, we recall that the Jones bosonic $q-$numbers [9, 1] are

$$[n]^V_{q^3,q} = \frac{q^{3n} - q^n}{q^3 - q}. \tag{27}$$

The recurrence relation for the Jones bosonic q-numbers (27)

$$[n+1]^V_{q^3,q} = (q^3 + q)[n]^V_{q^3,q} - q^4[n-1]^V_{q^3,q}$$

gives the "knot coefficients":

$$k^V_1 = q^3 + q, \; k^V_2 = q^4,$$

from which the "Link coefficients" (23), leading to (22), follow

$$l^V_2 = +(-k^V_2)^{\frac{1}{2}}, \quad l^V_1 = +(k^V_1 - 2l^V_2)^{\frac{1}{2}}.$$

6 HOMFLY (p, q)-numbers

The HOMFLY skein relation [10]

$$p^{-1}H_+(p,z) - pH_-(p,z) = zH_O(p,z) \tag{28}$$

defines the two-parameter HOMFLY polynomials $H(p,z)$. Let us make a change of the variable

$$z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}.$$

Rewriting (28) in the form (1),

$$H_+(p,q) = p(q^{\frac{1}{2}} - q^{-\frac{1}{2}})H_O(p,q) + p^2H_-(,q), \tag{29}$$

one obtains the "Link coefficients"

$$l^H_1 = p(q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad l^H_2 = p^2. \tag{30}$$

Then from

$$P + Q = p(q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad PQ = -p^2, \tag{31}$$

one has

$$P = pq^{\frac{1}{2}}, \quad Q = -pq^{-\frac{1}{2}}. \tag{32}$$

Thus, we obtain the HOMFLY fermionic $q-$numbers

$$[n]^{H}_{pq^{\frac{1}{2}},-pq^{-\frac{1}{2}}} = p^{n-1} \cdot \frac{(q^{\frac{1}{2}})^n - (-q^{-\frac{1}{2}})^n}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} = p^{n-1} \cdot [n]^{A}_{q^{\frac{1}{2}},-q^{-\frac{1}{2}}}. \tag{33}$$

And, at the end, we remind that the HOMFLY bosonic $q-$numbers look as

$$[n]^{H}_{p^2q,p^2q^{-1}} = p^{2(n-1)} \cdot \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{34}$$

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THE 1-2-3 EDGE LABELING AND VERTEX COLORING OF COMPLETE GRAPH K_N WITH A MODIFIED ALGORITHM

M.R. Farahani
Department of Mathematics
Iran University of Science and Technology
Tehran 16844, Iran
MR_Farahani@Mathdep.iust.ac.ir

S.H. Hosseini
Department of Mathematics
Iran University of Science and Technology
Tehran 16844, Iran
HamidHosseini@Mathdep.iust.ac.ir

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Abstract

In this paper, we introduce an algorithm which is able to create the 1-2-3-edge labeling of a complete graph Kn ($n \geq 3$). After running the algorithm we will obtain a function f: E(Kn) $\longrightarrow$ {1, 2, 3} such that we can classify the all complete graphs into three important classes K$3q$, K$3q$+1 and K$3q$+2. With running the modified edge labeling algorithm we can enumerate the number of edges with every label of {1, 2, 3}, in complete graph Kn ($n \geq 3$). Also, this algorithm is able to compute all S_i's for every i ($1 \leq i \leq n$). And for every classes we will denote very remarkable formulas which are useful for edge labeling and vertex coloring of all complete graphs.

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1. Introduction

In this paper we will present an algorithm which is able to create the 1-2-3-edge labeling of a complete graph K_n ($n \geqslant 3$), such that the sum of the edge labels for all vertices of complete graph are not equivalent for any two distinct vertices.

In other words for every complete graph $K_n = (V(K_n), E(K_n))$, there exist a function $f : E(K_n) \longrightarrow \{1, 2, \cdots, k\}$ be an edge labeling of the edges of a complete graph K_n. Also, for any two arbitrary vertices $u, v \in V(K_n)$ and edge $uv \in E(K_n)$ we will have $f(uv) \in \{1, 2, \cdots, k\}$. As a recall notice that the sum of the edge labels for all vertices is equal to $S_v := \sum_{u \in V(K_n)} f(uv)$. Therefore, for any two arbitrary distinct vertices $u, v \in V(K_n)$ we have $S_v \neq S_u$. If $k = 3$ then function f is a 1-2-3-edge labeling of complete graph K_n, such that we denote $f(uv)$ as label of edge uv and S_v as color of vertex v. It is easy to see that for every graph which has not a component, isomorphic to K_2, there exist such an edge labeling for some integer positive k.

In 2002, M. Karonski, et.al (Conjecture 1, see [6, 11]) conjectured that such a labeling with $k = 3$ is possible for all such graphs. And conjectured that there exist the 1,2 and 3 edge labeling and vertex coloring for complete graph K_n in [11]. Addario-Berry, et.al denote the first constant bound of k that is equal to 30 and proved it (see [1]), which was later improved to $k = 16$ in 2008 by Addario-Berry and et.al. [2] and $k = 13$ by T. Wang and Q. Yu in 2008, [14]. Also its new bounds are $k = 5$ and $k = 6$ by Kalkowski, et.al. [9, 10].

The purpose of introducing this approach is obtaining the modified algorithm for labeling complete graph K_n which is more faster than the ordering and labeling algorithms presented by M. Alaeiyan, M.R. Farahani and S. Mohammadian in [3]. Also, with running this algorithm we can reduce the computational complexity of ordering and labeling algorithms.

Our aim in this algorithm is to create the 1-2-3-edge labeling of complete graph K_n, very faster. To begin, we will use from the 1-2-3-edge labeling of complete graph K_3 such that with adding a new vertex x to K_3 and with labeling all new edges of graph we can create the 1-2-3-edge labeling of complete graph K_4. See Figure 1. By repeating this argument, at the end of algorithm we will construct the 1-2-3-edge labeling of complete graph K_n from the 1-2-3-edge labeling of complete graph K_{n-1} such that the sum of

the edge labels for all vertices of each complete graph are not equivalent. As a remark notice that with running this algorithm for the 1-2-3-edge labeling of complete graph K_n, we will obtain a function $f : E(K_n) \longrightarrow \{1, 2, 3\}$ such that we can partition the set $E(K_n)$ into three important sets E_n^3, E_n^2 and E_n^1. This is also the main contribution of this paper. To get a feeling for our approach we need to the following definitions.

Definition 1.1. *We define sets E_{n-1}^1,E_{n-1}^2 and E_{n-1}^3, as follows:*

$$E_{n-1}^1 = \{v_iv_j \in E(K_{n-1}) | f(v_iv_j) = 1\} \Rightarrow |E_{n-1}^1| = r_{n-1}$$

$$E_{n-1}^2 = \{v_iv_j \in E(K_{n-1}) | f(v_iv_j) = 2\} \Rightarrow |E_{n-1}^2| = b_{n-1}$$

$$E_{n-1}^3 = \{v_iv_j \in E(K_{n-1}) | f(v_iv_j) = 3\} \Rightarrow |E_{n-1}^3| = g_{n-1}.$$

Therefore $r_{n-1} + b_{n-1} + g_{n-1} = e_{n-1}$ such that $E_{n-1}^1 \cup E_{n-1}^2 \cup E_{n-1}^3 = E(K_{n-1})$ and $|E(K_{n-1})| = e_{n-1} = \frac{(n-1)(n-2)}{2}$.

Definition 1.2. *Let we have the 1-2-3-edge labeling of complete graph K_{n-1} next, we add a new vertex x to this graph for creating the 1-2-3-edge labeling of complete graph K_n in this case we define the new edges to sets B_n^1, B_n^2 and B_n^3 as follows:*

$$B_n^1 = \{xv_i \in E(K_n) | f(xv_i) = 1\} \Rightarrow |B_n^1| = \gamma_n$$

$$B_n^2 = \{xv_i \in E(K_n) | f(xv_i) = 2\} \Rightarrow |B_n^2| = \beta_n$$

$$B_n^3 = \{xv_i \in E(K_n) | f(xv_i) = 3\} \Rightarrow |B_n^3| = \alpha_n.$$

So, we have $n - 1 = \alpha_n + \beta_n + \gamma_n$ and also we will show that according to the algorithm we have $R_x := \sum_{v \in V(K_n - 1)} f(xv) = 3\alpha_n + 2\beta_n + \gamma_n$. The following theorem is the main result of this paper.

Theorem 1.3. *let complete graph $K_n = (V(K_n), E(K_n))$. Then*
(i) the 1-2-3-edge labeling of complete graph K_n will be obtain from the 1-2-3-edge labeling of complete graph K_{n-1} by a modified edge labeling algorithm such that this process start from the 1-2-3-edge labeling of complete graph K_3, where for each $u, v, w \in V(K_3)$ we have:

$$f(uv) = 3, f(vw) = 2, f(uw) = 1,$$

$$S_v = 5, S_u = 4, S_w = 3.$$

(ii) the sum of the edge labels for all vertices of complete graph K_n, *(*$S_i := \sum_{j=1,j\neq i}^{n} f(v_iv_j)$*) form a strictly descending sequence such that the maximum value of this sequence is equal to* S_1 *and the minimum value of sequence is equal to* S_n. *That means that for all* n, i $(n \geq 3, 1 \leq i \leq n)$ *we will have:*

$$n-1 < n \leq S_n < S_{n-1} < \cdots < S_2 < S_1 \leq 3(n-1)$$

Such that for all $(n > 6)$ $S_n = n+3$, $n+6 \geq S_{n-1} \geq n+5$, $n+12 \geq S_{n-2} \geq n+8$,...,$S_{2[\frac{n}{3}]} = 3n - 2[\frac{n}{3}] - 2$,...,$S_2 = 3n-4$ *and* $S_1 = 3n-3$. *Also this algorithm is finite and convergent to the acceptable solution.*

With running the modified edge labeling algorithm we can enumerate the number of edges with every label of $\{1, 2, 3\}$, in complete graph K_n $(n \geq 3)$. Also, this algorithm is able to compute all S_i's for every i $(1 \leq i \leq n)$. So, after proving the Theorem 1, the desire results can be viewed in the Section 4.

2. The modified edge labeling algorithm

In this section we introduce the modified algorithm for obtaining a functional edge labeling for all complete graphs K_n. In fact after the running algorithm we will obtain very remarkable formulas which are useful for edge labeling and vertex coloring of all complete graphs. Now it is time to introduce the algorithm. So the steps of the algorithm are as follows:

Step(0): at first we will start the 1-2-3-edge labeling of complete graph K_3 which is the first complete graph with this property. So, we consider complete graph $K_3 = (V(K_3), E(K_3))$ such that $V(K_3) = \{v_1, v_2, v_3\}$ and $E(K_3) = \{v_1v_2, v_1v_3, v_2v_3\}$. Let us assign weights to the edges of graph somehow that edge v_1v_2 get label 3, edge v_1v_3 get label 2 and edge v_2v_3 get label 1. Then we have vertices colors $S_1 = 5, S_2 = 4, S_3 = 3$.

Step(1): we add a vertex x to K_3 and all the connected edges (new edges) of vertex x get label 3. Then we compute the new sum of the edge labels for all vertices. Hence, we have $R_i = S_i + f(xv_i)$ for all i $(1 \leq i \leq 3)$ and $R_x := \sum_{v_i \in V(K_3)} f(xv_i)$. So $R_1 = 8$, $R_2 = 7$, $R_3 = 6$, $R_x = 9$.

It is easy to observe that none of the R_i's $(1 \leqslant i \leqslant 3)$ are not equivalent. This implies that, we can continue the algorithm without creating new conflicts. Then we order all R_i's ($i = 1, 2, 3$ and x) in descending form. By making such changes, we rename the all vertices of graph such that the

vertex with maximum value of R_i is v_1 with vertex color $S_1 = 9$ and the vertex with minimum value of R_i is v_4 with vertex color $S_4 = 7$. Also, analogously we name the another vertices. Here we have the 1-2-3-edge labeling of complete graph K_4. (See Figure 1).

Step(2): in this step of the algorithm we add a vertex x to the 1-2-3-edge labeling of complete graph K_4 and the all connected edges (new edges) of vertex x get label 3. Then we compute new sums of the edge labels for all vertices. So, we have $R_1 = 12$, $R_2 = 11$, $R_3 = 10$, $R_4 = 9$, $R_x = 12$. Since $R_x = 12 = R_1$, here there is a conflict for algorithm. Thus, for debugging the conflict, we must subtract one unit from the latest edge of vertex x which its label is not equal to 1. Then we will have $R_x = 11$ and $f(xv_4) = 2$. In this case since $R_x = 11 = R_2$, we go on as above. Then we have $R_x = 10$ and $f(xv_4) = 1$, again we have $R_x = R_3$. So, until the sums of any two distinct vertices are equivalent, we do as above. Then we subtract one unit from the latest edge of vertex x which is not the common edge of those vertices and also it has the largest index, thus $f(xv_2) := 2$, $R_2 = 10$ and $R_x = 9$. New, since $R_2 = R_3$. So, we must continue this algorithm. Hence, we subtract from the edge label xv_3 and $f(xv_3) := 2$. Then $R_x = 8$ and $R_3 = 9$. At last we have $f(xv_1) = 3$, $f(xv_2) = 2$, $f(xv_3) = 2$ and $f(xv_4) = 1$.

This process will continue until obtaining these values: $R_1 = 12, R_2 = 10$, $R_3 = 9$, $R_4 = 8$, $R_x = 7$. This is the acceptable solution of the algorithm for the 1-2-3-edge labeling of complete graph K_5 (See Figure 1).

Step$(n-3)$: Respectively in our final step we will create the 1-2-3-edge

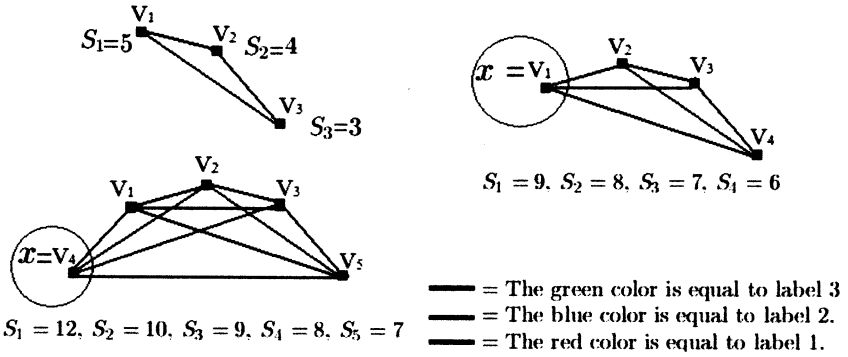


FIGURE 1. 1-2-3-edge labeling for graphs K_3, K_4 and K_5 by the modified algorithm of K_n

labeling of complete graph K_n by adding vertices to any complete graph in each step. Figure 2 is the intuition of step $n-3^{th}$.

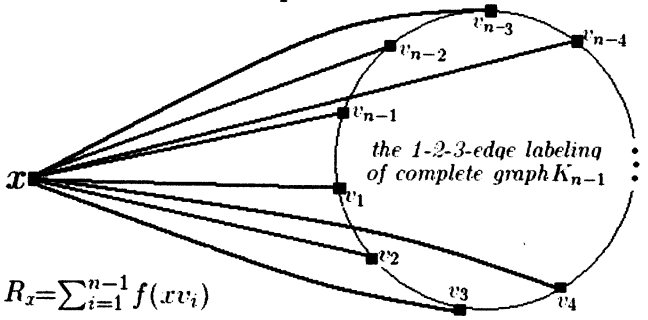


FIGURE 2. 1-2-3-edge labeling for graph K_n with adding a new vertex x to 1-2-3-edge labeling for graph K_{n-1} by the modified algorithm

3. PROOF OF THE THEOREM 1.3

Note that any complete graph K_n $(n \geqslant 3)$ follows from the Theorem 1 and will convergent to the acceptable solution. That means that, we will have the 1-2-3-edge labeling of complete graph K_n such that the sum of the edge labels for all vertices of graph are not equivalent. It is easy to show that these summations which will be obtained at the end of the algorithm form a strictly descending sequence such that the minimum value of the sequence S_n is more than $n-1$. This implies that, for the minimum value of the sequence S_n we can label at least $n-1$ external edges of graph with label 1. In this case there are no two vertices with equivalent labeled edges. For proving the Theorem 1, first we will need to prove the following lemma.

Lemma 3.1. *Let the 1-2-3-edge labeling of complete graph K_n $(n \geqslant 9)$ then for each new edge of graph with label 1 there exist two new edges with label 3.*

proof of Lemma 1. We consider the all new edges of complete graph K_n with set $B_n \subset E(K_n)=E(K_{n-1}) \bigcup\{xv_i|v_i \in V(K_{n-1})\}$. That means that, we can prove the Lemma 1 with adding the new labels to the old labels. First, for two vertices v_m, v_{m+1} $(m \in \mathbb{Z}_n = \{1,2,...,n\})$. We consider v_m. Now if $R_m = R_x$ then we have $R_x = \sum_{i=1,i\neq m,j}^{n-1} f(xv_i) + f(xv_m) + f(xv_j)$

and $R_m = S_m + f(xv_m)$, $(f(xv_m) = 3)$. Here, for debugging the created conflict the algorithm subtract one unit from the latest edge of complete graph K_n (from the edge xv_j with label 3, which j is maximum index). Then we have $f(xv_j) := 2$ and $R_x = \sum_{i=1,i\neq m,j}^{n-1} f(xv_i) + f(xv_m) + 2 \Rightarrow$ We will have $R_x := R_m - 1$.
Now we consider vertex v_{m+1}. If $R_{m+1} = R_x$, then similarly we do as above for edge xv_j, we will have $f(v_nv_j) = 1$. Hence, $R_x = \sum_{i=1,i\neq m,j}^{n-1} f(xv_i) + f(xv_m) + 1 \Rightarrow R_x := R_{m+1} - 1$.
This implies that for two edges xv_m, xv_{m+1} with labels 3 there exist one edge xv_j with label 1. Here, the proof of Lemma 1 was completed. ■
Note that according to the Definition 2, for any complete graph K_n $(n \geqslant 9)$, we will have $\alpha = 2\gamma$. Then for all new edges of complete graph K_n we have $n - 1 = 3\gamma + \beta$. Therefore, at last for R_x we will have:
$R_x = 3\alpha + 2\beta + \gamma = 7\gamma + 2\beta$ $(n = 3\gamma + \beta) \Rightarrow$ For $n > 4$; $R_n < R_x = 2n + \gamma < R_1$.

We are now equipped to prove the Theorem 1.
Proof of Theorem 1. for proving the theorem for complete graph K_n $(n \geqslant 3)$, we will use from the induction. By induction on n $(n \geqslant 3)$ we will have:
For $n = 3 \Rightarrow S_1 = 5 > S_2 = 4 > S_3 = 3 = n$ and $f(v_1v_2) = 3$, $f(v_1v_3) = 2$, $f(v_2v_3) = 1$. See Figure 1.
For $n = 4 \Rightarrow S_1 = 9 > S_2 = 8 > S_3 = 7 > S_4 = 6 = n + 2$ and $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_2v_1) = 3$, $f(v_2v_3) = 2$, $f(v_3v_4) = 1$. See Figure 1.
Now for $n = 5$ with similarly argument as above we create the 1-2-3-edge labeling of complete graph K_5 with summations $S_1 = 12 > S_2 = 10 > S_3 = 9 > S_4 = 8 > S_5 = 7 = n + 2$ such that $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = f(v_2v_3) = 3$, $f(v_2v_4) = f(v_2v_5) = f(v_3v_4) = 2$, $f(v_3v_5) = f(v_4v_5) = 1$. See Figure 1. And according to the modified algorithm, we obtain the 1-2-3-edge labeling of complete graph K_6 and K_7. See Figure 3 and Figure 4.
According to the previous section, after running the algorithm for every complete graph, it will obtain the sums which form a strictly descending sequence. Note that in each step, the algorithm after adding a new vertex x, change the notations of all vertices of graph such that after reordering. The

vertex with maximum summation will be equal to v_1 and the vertex with minimum summation will be equal to v_n. The another vertices of graph will name similarly corresponding to the sequence of sums such that for every $v_i \in E_n$ $(1 \leq i \leq n)$, the sum S_i is belong to the vertex v_i.

Now we consider $n = 6$ as the base of the induction and we will run the algorithm for the complete graph K_6. In order to the steps of the algorithm, at first we add a new vertex x to the 1-2-3-edge labeling of complete graph K_5. So, we will have:
$S_1 = 15 > S_2 = 14 > S_3 = 13 > S_4 = 12 > S_5 = 11 > S_6 = 9 = n + 3$ and $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = f(v_1v_6) = f(v_2v_3) = f(v_2v_4) = f(v_2v_5) = f(v_3v_4) = 3$, $f(v_2v_6) = f(v_3v_5) = f(v_3v_6) = f(v_4v_5) = 2$, $f(v_4v_6) = f(v_5v_6) = 1$.
Here, it is obvious that it follows from the Theorem 1. (See Figure 3). Now

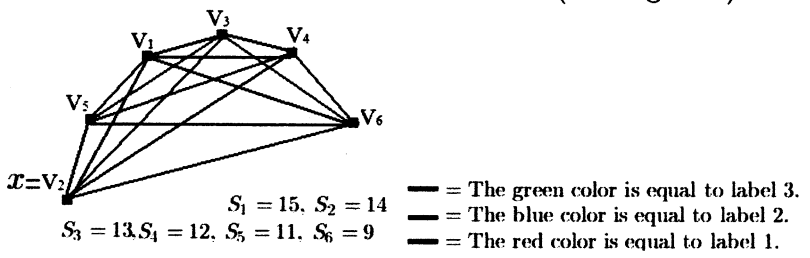

V3
V1
V4
V5
V6
x=V2
S1 = 15, S2 = 14
S3 = 13, S4 = 12, S5 = 11, S6 = 9
= The green color is equal to label 3.
= The blue color is equal to label 2.
= The red color is equal to label 1.

FIGURE 3. 1-2-3-edge labeling for graph K_6 by the modified algorithm

we suppose that the algorithm follows from Theorem 1 for the 1-2-3-edge labeling of complete graph K_{n-1} (the assumption of induction). Therefore, we will prove that there exist the 1-2-3-edge labeling of complete graph K_n. First, we add vertex x to the set $V(K_{n-1}) = \{v_1, ..., v_{n-1}\}$. Thus, we will have $V(K'_n) = \{v_1, ..., v_{n-1}, x\}$. Then we connect vertex x to all others vertices such that all edge labeled of complete graph K_{n-1} will be fixed until the end of the algorithm.
Now we must label $n - 1$ new edges of complete graph K_n in order to the algorithm. We assign label 3 to all new edges and then we compute all R_i's such that $R_i = S_i + f(xv_i)$ for every i $(1 \leq i \leq n - 1)$. Therefore $R_{n-1} = n + 5$, $n + 8 \geq R_{n-2} \geq n + 7$, $n + 13 \geq R_{n-3} \geq n + 10$,... and $R_1 = 3n - 3$.
Finally we will compute $R_x := \sum_{i=1}^{n-1} f(xv_i)$. Here, note that since x is

adjacent to v_1 then we have $R_1 = 3(n-1) = R_x$. So, we must subtract one unit from the label of edge xv_{n-1} (xv_{n-1} is the latest edge). This implies that $R_x \neq R_1$, $f(xv_{n-1}) = 2$ and $f(xv_1) = 3$. In other hands, since $R_{n-1} = S_{n-1} + 3$, then we will have $R_{n-1} = S_{n-1} + 3 - 1 = n + 4$ ($S_{n-1} = n + 2$).

Since vertex x is adjacent to vertex v_2 such that $f(xv_2) = 3$. So, if $R_2 = 3n - 4 = R_x$, then we must subtract one unit from the label of edge xv_{n-1} ($f(xv_{n-1}) = 2$). This implies that $R_x \neq R_2$. So, we have $R_{n-1} := n + 3$, $R_x = 3n - 5$ and $f(xv_{n-1}) = 1$. Again while $R_x = R_3$ then we must subtract one unit from the label of edge xv_{n-2} ($f(xv_{n-2}) = 3$), since $f(xv_n) = 1$. So $f(xv_{n-2}) := 2$. Then we will have $R_3 = 3n - 5$, $R_{n-2} := R_{n-2} - 1 \geq n + 4 + 2 = n + 6$ and $R_3 > R_x = 3n - 6$. Now while $R_x = R_4$ then we do as above.

Generally, for any arbitrary vertex v_i which follows from $R_x = R_i$ we subtract one unit from the label of the latest edge which is not the common edge of those vertices. As a remark notice that its label is opposed to 1. In other words, we have:

If $\exists i \in \mathbb{Z}_{n-1}$ such that $R_x = R_i \Longrightarrow \exists m \neq i$ such that $f(xv_m) \neq 1$ and $\forall j$ ($m < j \leq n-1$), $f(xv_j) = 1$. So, we will have $f(xv_m) := f(xv_m) - 1$.

As a remark notice that while for $n = 3q + 2$ ($q \geq 2$), $R_x = R_i$ such that $\forall j$ ($i < j \leq n-1$) we have $f(xv_j) = 1$ and $f(xv_i)f(xv_{i-1}) = 3$. Therefore $\exists m$ $m = i - 1$, $f(xv_{i-1}) := f(xv_{i-1}) - 1 = 2$ and $R_x < R_i$ but $R_{i-1} = R_i$. Then for debugging the conflict $R_i = R_{i-1}$, we must subtract one unit from the label edge xv_i, because $\forall j$ ($i + 1 \leq j \leq n$), $f(xv_j) = 1$. Hence, we will have $f(xv_i) = f(xv_{i+1}) = 2$, $R_x := R_{i+1} - 1$ and $R_{i+1} := R_i - 1$. In other words, $R_x < R_i < R_{i-1}$. For $n = 3q, 3q + 1$ we proved by Lemma 1. This implies that according to the 1-2-3-edge labeling of complete graph K_{n-1} for vertex x ($R_x = 3\alpha_n + 2\beta_n + \gamma_n$)we have several possible solutions at the end of the algorithm. Since there exist the number of γ_n edges with label 1 and the number of $2\gamma_n$ edges with label 3. Here, in every time we can achieve to the latest vertex of graph. This implies that the algorithm is convergent.

We proved that without changing the labeling of the 1-2-3-edge labeling of complete graph K_{n-1}, the algorithm label all new edges of graph and compute all new R_i's ($i = 1, ..., n-1, x$) which obeys the conditions of Theorem 1.

Now it is enough to prove that the algorithm in each case is convergent.

First according to the assumption of induction, we have the 1-2-3-edge labeling of complete graph K_{n-1} ($\forall v > 6$). So, for every vertex $v_i \in V(K_{n-1})$ $(1 \leq i \leq n-1)$, we can compute all R_i's such that for two vertices v_1, v_{n-1}. Therefore, there exists $R_1 = 3n-6+f(xv_1)$ and $R_{n-1} = n+2+f(xv_{n-1})$. This implies that we have the number of $3n-6-(n-1)-(n-1) = n-4$ sums for vertex x.

Now we want to assume a very bad condition on edge labeling of graph such that we have the maximum number of conflicts on algorithm. In such a very bad case we assume $f(xv_i) = 3$, $1 \leq i \leq 4$ (according to the above mentioned) and we assign the number of $n-5$ labels 1 to another new edges of graph. Therefore, we obtain $R_x = f(xv_1) + f(xv_2) + ... + f(xv_{n-1}) = 3+3+3+3+n-5 = n+7$. That means that according to the assumption of induction, we have $\forall n > 6$ $S_{n-1} = n+2$, $n+6 \geq S_{n-2} \geq n+5$ and $n+12 \geq S_{n-3} \geq n+8$. For $n = 3, 4, 5$ and 6, see Figure 1, 3. Hence, $R_{n-3} \geq n+7+f(xv_{n-3}) > R_x = n+7 > n+5+f(xv_{n-2}) \geq R_{n-2} \geq n+4+f(xv_{n-2})$ then $n+8 > n+7 > n+6$. rhus, the algorithm in very bad conditions will be convergence.

In addition according to the end of algorithm for vertex v_n, we have $f(v_1v_n) = 3$, $f(v_2v_n) = f(v_3v_n) = 2$ and $f(v_iv_n) = (v_iv_n) = 1$ $(4 \leq i \leq n-1)$. Therefore, $R_{n-1} = f(xv_{n-1}) + \sum_{i=1}^{n-2} f(v_iv_{n-1}) \Longrightarrow S_n = \sum_{i=1}^{n-1} f(v_iv_n) = 3+2+2+n-4 = n+3$. That means that, in the end of algorithm $S_n = n+3$ 9or R_{n-1}). This implies that the requisite for convergence of algorithm is obtaining the acceptable solution. Finally we achieved the 1-2-3-edge labeling of complete graph K_n. Here, the proof of the theorem was finished. ■

We are now able to use the Theorem 1 for classifying all complete graphs K_n into three family K_{3q}, K_{3q+1} and K_{3q+2} (recall that the value β is equal to 0 or 1 or 2). Hence, we will denote many different properties as the important results for all complete graphs in these families. We will present sum R_x for new vertex x which was added to the 1-2-3-edge labeling of complete graph K_{n-1} and we can enumerate the number of new edges with any labels of $\{1, 2, 3\}$, also we recognize the labeled edges with any labels of $\{1, 2, 3\}$ in each family.

At last we will denote the important formulas for obtaining all S_i's $(1 \leq i \leq n)$ for any family of the complete graphs.

Note that in this case we can use colors (red, blue, green) instead of labels $\{1, 2, 3\}$ respectively. That is because that we can use from these color

graphs in modeling related path among the different users. So that we can cover this network with labeling all the paths (one-way paths have label 1, two-way paths have label 2 and three-way paths have label 3). So, we distinguish among the different paths with special characteristics.

4. Conclusions and Classification of K_n

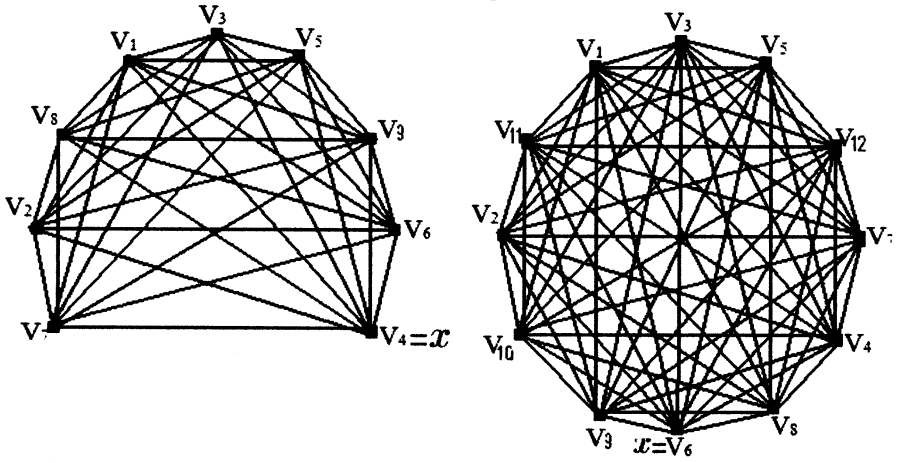
Let we have the 1-2-3-edge labeling of complete graph $K_{n-1} = (V(K_{n-1}), E(K_{n-1}))$ such that $|V(K_{n-1})| = n-1$, $|E(K_{n-1})| = \frac{(n-1)(n-2)}{2}$. We consider $K_n = (V(K_n), E(K_n))$ such that $V(K_n) = V(K_{n-1}) \bigcup \{x\}$, $E(K_n) = E(K_{n-1}) \bigcup B_n$ and $B_n = \{xv_i | v_i \in V(K_{n-1})\}$. We assume that vertex x is the new added vertex to K_{n-1} in the $n-3^{th}$ step of the modified algorithm. Now according to the Definition 1 and Definition 2 in previous sections, it is easy to show that the defined sets obeys the following recursive formulas for all integer n ($n \geq 3$).

$r_n = |E_n^1| = |E_{n-1}^1| + \gamma_n$, $b_n = |E_n^2| = |E_{n-1}^2| + \beta_n$ and $g_n = |E_n^3| = |E_{n-1}^3| + \alpha_n$; such that $\alpha_4 = 3, \beta_4 = \gamma_4 = 0$, $|E_3^1| = |E_3^2| = |E_3^3| = 1$.
It is obvious that we can partition the set $E(K_n)$ into three important subsets. Also $E_n^1 = E_{n-1}^1 \bigcup B_n^1$, $E_n^2 = E_{n-1}^2 \bigcup B_n^2$ and $E_n^3 = E_{n-1}^3 \bigcup B_n^3$ such that $B_n = B_n^3 \bigcup B_n^2 \bigcup B_n^1$.
Thus, it is time to classify all complete graphs into three classes K_{3q}, K_{3q+1} and K_{3q+2} ($q \geq 2$) according to the algorithm and the Theorem 1. Here, we avoid to denote the results which can obtain from drawing graphs.

4.1. **Class of K_{3q}.** The results of running algorithm for the 1-2-3-edge labeling of complete graph K_n; $n = 3q$. According to the algorithm and Figure 4 of graph $K_{3q} = K_{3q-1} \bigcup \{x\}$, we will have the following sets and results:

$B_n^3 = \{xv_i \in E(K_{3q}) | i = 1, 2, ..., 2q\} \Rightarrow |B_n^3| = \alpha_{3q} = \mathbf{2q}$.
$B_n^2 = \{xv_{2q+1} \in E(K_{3q})\} \Rightarrow |B_n^2| = \beta_{3q} = \mathbf{1}$.
$B_n^1 = \{xv_i \in E(K_{3q}) | i = 2q+2, ..., 3q-1\} \Rightarrow |B_n^1| = \gamma_{3q} = n-2q-1 = \mathbf{q-2}$.
So we have $E(K_{3q}) = E_n^3 \bigcup E_n^2 \bigcup E_n^1$ such that $|E_n^k| = |E_{n-1}^k| + |B_n^k|$ ($k = 1, 2, 3$).
Now with adding all new edges, we can create the 1-2-3-edge labeling of complete graph K_{3q}. Therefore, it is time to compute all S_i's. But notice

that in this case, according to the algorithm in the 1-2-3-edge labeling of complete graph K_{3q}, vertex x (with $R_x = 7q$) change to the vertex v_{2q-2}. Therefore, summations on $2q+1$ vertices of graph obeys from the following formula:
$\forall i\ (1 \leq i \leq 2q+1); S_{i+1} = S_i - 1$ such that $S_1 = 9q-3, S_{2q+1} = 7q-3$.
In addition, summations on another vertices of graph are listed for large n as follows:
$S_n = n+3 = 3q+3$, $S_{n-1} = S_n + 2 = 3q+5$, $S_{n-2} = S_{n-1} + 4 = 3q+9$, $S_{n-3} = S_{n-2} + 5 = 3q+14$, $S_{n-4} = S_{n-3} + 4 = 3q+18$, $S_{n-5} = S_{n-4} + 3 = 3q+19$, $S_{n-6} = S_{n-5} + 3 = 3q+21$,..., $S_{2q+2} = S_{2q+1} - 3 = 7q-6$.



$S_1 = 24$, $S_2 = 23$, $S_3 = 22$,
$S_4 = 21$, $S_5 = 20$, $S_6 = 19$,
$S_7 = 18$, $S_8 = 14$, $S_9 = 12$,

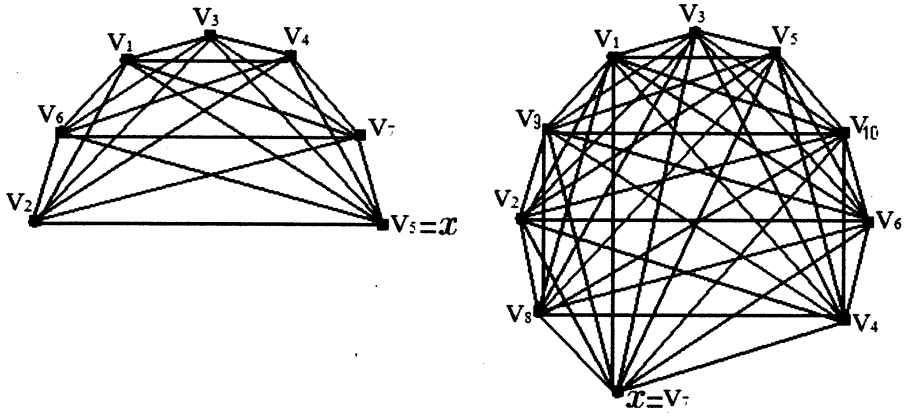
$S_1 = 33$, $S_2 = 32$, $S_3 = 31$, $S_4 = 30$
$S_5 = 29$, $S_6 = 28$, $S_7 = 27$, $S_8 = 26$
$S_9 = 25$, $S_{10} = 21$, $S_{11} = 17$, $S_{12} = 15$

FIGURE 4. The example of 1-2-3-edge labeling for class of K_{3q} by the modified algorithm.

4.2. **Class of K_{3q+1}.** The results of running algorithm for the 1-2-3-edge labeling of complete graph K_n; $n = 3q+1$. According to the algorithm and Figure 5 of graph, we will have the following results:

$B_n^3 = \{xv_i \in E(K_{3q+1}) | i = 1, \ldots, 2q\} \Rightarrow \alpha_{3q+1} = \mathbf{2q}$.
$B_n^2 = \emptyset \Rightarrow \beta_{3q+1} = \mathbf{0}$.
$B_n^1 = \{xv_j \in E(K_{3q+1}) | j = 2q+1, \ldots, 3q\} \Rightarrow \gamma_{3q+1} = \mathbf{q}$.

Now with adding all new edges to the 1-2-3-edge labeling of complete graph K_{3q}, we can create the 1-2-3-edge labeling of complete graph K_{3q+1}. Therefore, it is time to compute all S_i's. But notice that in this case, according to the algorithm in the 1-2-3-edge labeling of complete graph K_{3q+1}, vertex x (with $R_x = 7q$) change to the vertex v_{2q+1}. Therefore, summations on $2q+1$ vertices of graph obeys from the following formula:
$\forall i\ (1 \leq i \leq 2q+1)$; $S_{i+1} = S_i - 1$ such that $S_1 = 9q$, $S_{2q+1} = 7q$.
In addition, summations on another vertices of graph K_{3q}, for large q are listed as follows:
$S_n = n+3 = 3q+4$, $S_{n-1} = S_n+2 = 3q+6$, $S_{n-2} = S_{n-1}+4 = 3q+10$, $S_{n-3} = S_{n-2}+5 = 3q+15$, $S_{n-4} = S_{n-3}+4 = 3q+19$, $S_{n-5} = S_{n-4}+5 = 3q+24$, $S_{n-6} = S_{n-5}+4 = 3q+28, \ldots$, $S_{2q+3} = S_{2q+2} - 3$ and $S_{2q+2} = S_{2q+1} - 2$.



$S_1 = 18$, $S_2 = 17$, $S_3 = 16$, $S_4 = 15$ $S_5 = 14$, $S_6 = 12$, $S_7 = 10$

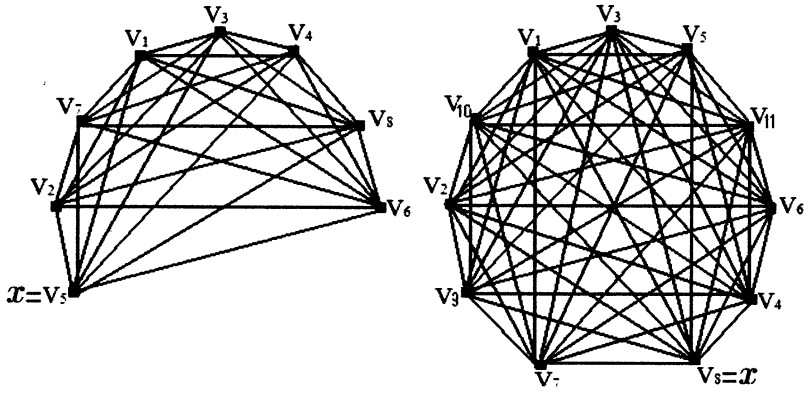
$S_1 = 27$, $S_2 = 26$, $S_3 = 25$, $S_4 = 24$, $S_5 = 23$ $S_6 = 22$, $S_7 = 21$, $S_8 = 18$, $S_9 = 15$, $S_{10} = 13$

FIGURE 5. The example of 1-2-3-edge labeling for class of K_{3q+1} by the modified algorithm

4.3. **Class of K_{3q+2}.** The results of running algorithm for the 1-2-3-edge labeling of complete graph K_n; $n = 3q+2$. According to the algorithm and Figure 6 of this graph. We will have the following results:

$B_n^3 = \{xv_i \in E(K_{3q+2}) | i = 1, \ldots, 2q-1\} \Rightarrow \alpha_{3q+2} = \mathbf{2q-1}$.
$B_n^2 = \{xv_{2q}, xv_{2q+1} \in E(K_{3q+2})\} \Rightarrow \beta_{3q+2} = \mathbf{2}$.
$B_n^1 = \{xv_j \in E(K_{3q+2}) | j = 2q+2, \ldots, 3q+1\} \Rightarrow \gamma_{3q+2} = \mathbf{q}$.

Now with adding all new edges to the 1-2-3-edge labeling of complete graph K_{3q+1}, we can create the 1-2-3-edge labeling of complete graph K_{3q+2}. Therefore, it is time to compute all S_i's. But notice that in this case, according to the algorithm in the 1-2-3-edge labeling of complete graph K_{3q+2}, vertex x (with $R_x = 7q+1$) change to the vertex v_{2q+2}. Therefore, summations on $2q+1$ vertices of graph obeys from the following formula: $\forall i\ (1 \leq i \leq 2q+1)$; $S_{i+1} := S_i - 1$ such that $S_1 = 9q+3$. As a remark notice that we have a exception for vertex v_{2q} such that $S_{2q} = S_{2q-1} - 2$. That means that we have for large q: $S_1 = 9q+3$, $S_2 = 9q+2, \cdots, S_{2q-1} = 7q+5$, $S_{2q} = 7q+3$, $S_{2q+1} = 7q+2$, $S_{2q+2} = 7q+1$.
In addition, summations on another vertices of graph are listed as follows: $S_n = n+3 = 3q+5$, $S_{n-1} = S_n+2 = 3q+7$, $S_{n-2} = S_{n-1}+4 = 3q+11$, $S_{n-3} = S_{n-2}+5 = 3q+16$, $S_{n-4} = S_{n-3}+4 = 3q+20$, $S_{n-5} = S_{n-4}+4 = 3q+24$, $S_{n-6} = S_{n-5}+4 = 3q+28, \cdots$, $S_{2q+4} = S_{2q+3} - 4$ and $S_{2q+3} = S_{2q+4} - 3$.



$S_1 = 21,\ S_2 = 20,\ S_3 = 19,\ S_4 = 17$
$S_5 = 16,\ S_6 = 15,\ S_7 = 13,\ S_8 = 11$

$S_1 = 30,\ S_2 = 29,\ S_3 = 28,\ S_4 = 27$
$S_5 = 26,\ S_6 = 24,\ S_7 = 23,\ S_8 = 22$
$S_9 = 19,\ S_{10} = 14,\ S_{11} = 12$

FIGURE 6. The example of 1-2-3-edge labeling for class of K_{3q+2} by the modified algorithm

4.4. **Enumerating the number of labeled edges.** At last we can enumerate the number of labeled edges for the 1-2-3-edge labeling of complete graph K_n $(n \geq 6)$ with using all denoted results.

First, the number of labeled edges with label 3:
For $n = 3q$ and $q \geq 2$, we have $g_5 = 5$ and $|E_n^3| = |E_{n-1}^3| + |B_n^3|$.
$g_n = g_5 + \sum_{i=6}^{n=3q} \alpha_i = g_5 + \sum_{i=2}^{q-1} \alpha_{3i+2} + \sum_{i=2}^{q-1} \alpha_{3i+1} + \sum_{i=2}^{q} \alpha_{3i} = 5 + \sum_{i=2}^{q-1} (2i-1) + \sum_{i=2}^{q-1} 2i + \sum_{i=2}^{q} 2i$. Thus,

$$\mathbf{g_{3q} = 3q^2 - 2q + 1}$$

Now for $n = 3q+1$ and $q \geq 2$, we have: $g_{3q+1} = g_{3q} + \alpha_{3q+1} = 3q^2 - 2q + 1 + 2q$. Thus,

$$\mathbf{g_{3q+1} = 3q^2 + 1}$$

Now for $n = 3q + 2$ and $q \geq 2$: $g_{3q+2} = g_{3q+1} + \alpha_{3q+2} = 3q^2 + 1 + 2q - 1$. Thus,

$$\mathbf{g_{3q+2} = 3q^2 + 2q}$$

The latter, the number of labeled edges with label 2:
For $n = 3q$ and $q \geq 2$, we have $b_5 = 3$ and $|E_n^2| = |E_{n-1}^2| + |B_n^2|$.
$b_n = b_5 + \sum_{i=6}^{n=3q} \beta_i = b_5 + \sum_{i=2}^{q-1} \beta_{3i+2} + \sum_{i=2}^{q-1} \beta_{3i+1} + \sum_{i=2}^{q} \beta_{3i} = 3 + \sum_{i=2}^{q-1} 2 + \sum_{i=2}^{q-1} 0 + \sum_{i=2}^{q} 1$. Thus,

$$\mathbf{b_{3q} = 3q - 2}$$

Now for $n = 3q + 1$ and $q \geq 2$: $b_{3q+1} = b_{3q} + \beta_{3q+1} = 2q - 3 + 0$. Thus,

$$\mathbf{b_{3q+1} = 3q - 2}$$

Now for $n = 3q + 2$ and $q \geq 2$: $b_{3q+2} = b_{3q+1} + \beta_{3q+2} = 2q - 3 + 2$. Thus,

$$\mathbf{b_{3q+2} = 3q}$$

At last, the number of labeled edges with label 1:
For $n = 3q$ and $q \geq 2$, we have $r_5 = 2$ and $|E_n^1| = |E_{n-1}^1| + |B_n^1|$.
$r_n = r_5 + \sum_{i=6}^{n=3q} \gamma_i = r_5 + \sum_{i=2}^{q-1} \gamma_{3i+2} + \sum_{i=2}^{q-1} \gamma_{3i+1} + \sum_{i=2}^{q} \gamma_{3i} = 2 + \sum_{i=2}^{q-1} i + \sum_{i=2}^{q-1} i + \sum_{i=2}^{q} (i-2) = \frac{q^2-3q+2+2q^2-2q-4}{2} + 2 = \frac{3q^2-5q+2}{2}$. Thus,

$$\mathbf{r_{3q} = 2q - 3.}$$

Now for $n = 3q + 1$ and $q \geq 2$: $|E_n^1| = r_{3q+1} = r_{3q} + \gamma_{3q+1} = \frac{3q^2-5q+2}{2} + q$. Thus,

$$\mathbf{r_{3q+1} = \frac{3q^2 - 3q + 2}{2}.}$$

Now for $n = 3q + 2$ and $q \geq 2$: $|E_n^1| = r_{3q+2} = r_{3q+1} + \gamma_{3q+2} = \frac{3q^2-3q+2}{2} + q$. Thus,

$$\mathbf{r_{3q+2} = \frac{3q^2 - q + 2}{2}}.$$

5. Problems

In our final section we want involve the reader with collecting the most intriguing conjectures in the topic as further researches. Maybe some of them could be solved in the near future.

Conjecture 1. *(The 1-2-3-conjecture [3, 6, 11]) Every connected graph $G = (V, E)$ non-isomorph to K_2 (with at least two edges) has an edge labeling $f : E \longrightarrow \{1, 2, 3\}$ and vertex coloring $S : V \longrightarrow \{n - 1, ..., 3n - 3\}$.*

Conjecture 2. *(n vertex coloring, [3]) There are distinct numbers of S_v's, $v \in V(G)$, of a graph G of order n, for a 1-2-3-edge labeling and vertex coloring.*

Conjecture 3. *(Proper vertex coloring, [3]) For all graph G of order n, there are the $\chi(G)$ numbers of S_v's, $v \in V(G)$, with this 1-2-3-edge labeling and vertex Coloring. Where $\chi(G)$ is the number colors of the vertices on the graph G.*

In this parer, we show that the complete graph K_n ($n \geq 3$), recognize in three conjecture 1, 2 and 3.

Conjecture 4. *(The 1,2-conjecture [8, 12, 13]) Every graph G=(V,E) has a coloring chip configuration $c : V \bigcup E \longrightarrow \{1, 2\}$.*

Conjecture 5. *(Antimagic labeling [4, 7]) For every connected graph G=(V,E) (with at least two edges) there is a bijection $c : E \longrightarrow \{1, 2, \cdots, |E|\}$ such that no two vertices of G have the same potential.*

Conjecture 6. *(Lucky labeling, [4, 5]) For every graph $G = (V; E)$ there is a vertex labeling $c : V \to \{1, 2, ..., \chi(G)\}$, whose vertex potential $q_v = \sum_{u \in N(v)} c(u)$ is a proper coloring of G.*

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DEPARTMENT OF MATHEMATICS, IRAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NARMAK, TEHRAN 16844, IRAN,

E-mail address: MR_Farahani@Mathdep.iust.ac.ir

DEPARTMENT OF MATHEMATICS, IRAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NARMAK, TEHRAN 16844, IRAN,

E-mail address: HamidHosseini@Mathdep.iust.ac.ir

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GENERIC MODELS IN ENSEMBLE OF SCALAR FIELD COSMOLOGY – TOWARD SELECTION OF A FORM OF POTENTIAL OF SCALAR FIELD[1]

Marek Szydlowski
Astronomical Observatory
Jagiellonian University
Orla 171, 30-244 Krakow, Poland

Mark Kac Complex Systems Research Centre
Jagiellonian University
Reymonta 4, 30-059 Krakow, Poland
marek.szydlowski@uj.edi.pl

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Abstract

We define space of all dynamical systems representing evolution of homogeneous cosmological models with scalar field as the Banach space with the Sobolev metric. In scalar field cosmology appears the problem of undetermination of the potential of the scalar field. Which potential for the scalar field should be chosen? From the latest Planck measurements some class potential with flat plateau is favored. On other hand we present theoretical arguments that the cosmological model should be generic rather than fine tuned. We are looking for generic class of dynamical systems in the ensemble for selection of a suitable form of the potential of the scalar field. For this aim we explore framework of structural stability and Peixoto theorem.

APS 2000 Physics subject classification 98.80.-k;

Keywords: scalar field cosmology, dynamical systems, structural stability

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1 Introduction

Main topics covered in this presentation are:

- scalar field cosmology (SFC) as a dynamical system,
- ensemble of SFC,
- structural stability issues,
- toward searching for potential of cosmological scalar field from structural stability.

Cosmology is the effective theory of the Universe at the large structure. These are characteristic properties of modern cosmology:

- It is based on the general relativity theory which is the effective theory of the gravity.
- Classical cosmology possesses cut-offs (works beyond the Planck scales).
- The SCM or ΛCDM model uses in modeling the Universe some parameters like density parameters.
- The cosmological constant parameter Λ describes the dark energy and dark matter sector in the form of pressureless matter $p = 0$.
- In this scheme the cosmological constant parameter is only the effective description of dark sector and we believe that the ΛCDM model emerges from a more fundamental theory.

The matter content of the Universe is described in terms of the barotropic equation of state — the relation between the pressure p and energy density ρ: $p = p(\rho)$. In general relativity and cosmology matter is modeled by the energy-momentum tensor $T_{\mu\nu}$ which covariant derivative is zero because of the Bianchi identity.

From the cosmological principle, i.e. the assumption of homogeneity and isotropy of the geometry as well as matter at the large scaling structure of the Universe, matter has the form of perfect fluid.

The condition of vanishing divergence of $T^{\mu\nu}{}_{;\mu}$ gives for the FRW model (which satisfies the cosmological principle) the simple relation

$$\frac{d\rho}{dt} = -3H(\rho + p), \tag{1.1}$$

where $H = (\ln a)^{\cdot}$, $a(t)$ is the scale factor, H is the Hubble function, t is the cosmological time

From the Einstein equation for homogeneous and isotropic space we obtain acceleration equation in the form

$$\frac{\ddot{a}}{a} = -\frac{1}{6}[\rho_{\text{eff}}(a) + 3p_{\text{eff}}(a)], \tag{1.2}$$

where ρ_{eff} and p_{eff} are effective energy density and pressure respectively; here and after we assume $8\pi G = c = 1$ (natural system of units). For the ΛCDM model ρ_{eff} and p_{eff} assume the form

$$\rho_{\text{eff}} = \rho_{m,0}a^{-3} + \Lambda \tag{1.3a}$$
$$p_{\text{eff}} = 0 - \Lambda. \tag{1.3b}$$
$$\tag{1.3c}$$

The acceleration equation (1.2) can be rewritten to the form

$$\ddot{a} = -\frac{\partial V}{\partial a}, \tag{1.4}$$

where $V = V(a) = -\frac{\rho_{\text{eff}}a^2}{6}$.

In cosmology it is convenient to use dimensionless parameters, density parameters

$$\Omega_i = \frac{\rho_i}{3H_0^2}, \tag{1.5}$$

where $\rho_{\text{crit}} = 3H_0^2$ is the critical value of energy density for a flat model.

In the ΛCDM model they are

$$\begin{gathered}\Omega_m = \frac{\rho_m}{3H_0^2} = \frac{\rho_{m,0}a^{-3}}{3H_0^2} = \Omega_{m,0}a^{-3} = \Omega_{m,0}(1+z)^3, \\ \Omega_\Lambda = \frac{\rho_\Lambda}{3H_0^2} = \frac{\Lambda}{3H_0^2} = \Omega_{\Lambda,0},\end{gathered} \tag{1.6}$$

where z is redshift $(1+z=a^{-1})$ and index zero denote the quantities which we measure at the present epoch.

After time reparametrization following $t \to \tau : \ |H_0|dt = d\tau$ we obtain that evolution of the Universe is given in the form of 2D dynamical system

$$\frac{dx}{d\tau} = y, \tag{1.7a}$$

$$\frac{dy}{d\tau} = -\frac{\partial V}{\partial x}(x), \tag{1.7b}$$

where $x = \frac{a}{a_0}$; a_0 is the present value of the scale factor.

System (1.7) has a first integral (called Friedmann equation)

$$\frac{y^2}{2} + V(x) = \frac{1}{2}\Omega_{k,o}, \tag{1.8}$$

where $\Omega_{k,0}$ is density parameters for curvature;

$$\Omega_k = \frac{\rho_k}{3H_0^2} = -\frac{3k}{3H_0^2}a^{-2} = \Omega_{k,0}a^{-2}. \tag{1.9}$$

Different forms of potential of cosmological models with dark energy and modified FRW equations are completed in Table I and Table II (for statistical comparison of these classes of cosmological mmodels see [1, 2]).

Main advantage of using dynamical systems in the context of cosmology, where we do not know initial condition for the Universe, is possibility of investigation of all possible evolutional scenarios for all possible initial conditions.

Definition 1.1 *The multiverse of cosmological models of accelerating Universe is the space of all 2-dimensional systems of the Newtonian type $\dot{x} = y$, $\dot{y} = -\partial V/\partial x$ with a suitably defined potential function of the scale factor, which characterizes the physical model of the dark energy or modification of the FRW equation.*

For different potential form $V(a)$ we obtain different phase portraits on the phase plane $(x,\ \dot{x})$. Moreover the Friedmann first integral can be used to classify all evolutional scenarios in the configuration space. In this schema the energy levels determined by $\Omega_{k,0}$ represent model evolution.

Table 1: The potential functions for different dark energy models.

model	potential function	parameters
Einstein-de Sitter model $\Omega_{\mathrm{m},0}=1$, $\Omega_{\Lambda,0}=0$, $\Omega_{k,0}=0$	$V(x)=-\frac{1}{2}\Omega_{\mathrm{m},0}x^{-1}$	H_0
ΛCDM model $\Omega_{\mathrm{m},0}+\Omega_{\Lambda,0}=1$	$V(x)=-\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1}+\Omega_{\Lambda,0}x^{2}\right)$	H_0, $\Omega_{\mathrm{m},0}$
FRW model filled with n noninteracting multi-fluids $p=w\rho$ with dust matter	$V(x)=-\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\sum_{i=1}^{N}\Omega_{i,0}x^{-3(1+w_i)}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\Lambda,0}$, $\Omega_{n,0}$
FRW quintessence model with dust and dark matter X $w_X<-1$ phantom models	$V(x)=-\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\Omega_{X,0}x^{-1-3w_X}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{X,0}$
FRW model with generalized Chaplygin gas [3, 4] $p=-\frac{A}{\rho^{\alpha}}$, $A>0$	$V(x)=-\frac{1}{2}\left[\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\Omega_{\mathrm{Chapl},0}\left(A_S+\frac{1-A_S}{x^{3(1+\alpha)}}\right)^{\frac{1}{1+\alpha}}\right]$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{Chapl},0}$
FRW models with dynamical equation of state for dark energy $p_X=w_X(a)\rho_X$ and dust	$V(x)=-\frac{1}{2}\left[\Omega_{\mathrm{m},0}x^{-1}+\Omega_{k,0}+\Omega_{X,0}x^{-1}\exp\left(\int_1^x\frac{w_X(a)}{a}da\right)\right]$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{X,0}$
FRW models with dynamical equation of state for dark energy coefficient equation of state $w_X=w_0+w_1z$	$V(z)=-\frac{1}{2}\left[\Omega_{\mathrm{m},0}(1+z)+\Omega_{X,0}(1+z)^{1+3(w_0-w_1)}e^{3w_1z}+\Omega_{k,0}\right]$	H_0, $\Omega_{X,0}$, w_0, w_1

Table 2: The potential functions for different cosmological models which offer the possibility of explanation of acceleration in terms of modified FRW equations.

model	potential function	parameters
non-flat Cardassian models filled by dust matter [5, 6]	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{k,0} + \Omega_{\mathrm{Card},0}x^{m+2}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{Card},0}$
bouncing models $(H/H_0)^2 = \Omega_{\mathrm{m},0}x^{-m}$, $-\Omega_{n,0}x^{-n}, n > m$ [7]	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-m+2} - \Omega_{n,0}x^{-n+2}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{n,0}$, m, n
Randall-Sundrum brane models with dust on the brane and dark radiation [8]	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\lambda,0}x^{-6} + \Omega_{k,0} + \Omega_{d,0}x^{-4}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\lambda,}0$
cosmology with spin and dust (MAG cosmology) [9]	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{s,0}x^{-6} + \Omega_{k,0}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{k,0}$
Dvali, Deffayet, Gabadadze brane models (DDG) [10]	$V(x) = -\frac{1}{2}\left(\sqrt{\Omega_{\mathrm{m},0}x^{-1} + \Omega_{rc,0}} + \sqrt{\Omega_{rc,0}}\right)^2$	H_0, $\Omega_{\mathrm{m},0}$
Sahni, Shtanov brane models [11, 12]	$V(x) = -\frac{1}{2}\left(\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\sigma,0} + 2\Omega_{l,0} \pm 2\sqrt{\Omega_{l,0}}\sqrt{\Omega_{\mathrm{m},0}x^{-1} + \Omega_{\sigma,0} + \Omega_{l,0} + \Omega_{\Lambda_b,0}}\right)$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\sigma,0}$, $\Omega_{l,0}$, $\Omega_{\Lambda_b,0}$
FRW models of nonlinear gravity with matter $\mathcal{L} \propto R^n$ and radiation [13, 14]	$V(x) = -\frac{1}{2}\left[\frac{2n}{3-n}\Omega_{\mathrm{m},0}x^{-1} + \frac{4n(2-n)}{(n-3)^2}\Omega_{r,0}x^{-2}\right]\Omega_{\mathrm{nonl},0}x^{\frac{3(n-1)}{n}}$	H_0, $\Omega_{\mathrm{m},0}$, $\Omega_{\mathrm{nonl},0}$
ΛDGP model [15, 16] screened cosmological constant model	$V(x) = -\frac{1}{8}x^2\left[-\frac{1}{r_0H_0} + \sqrt{\left(2 + \frac{1}{r_0H_0}\right)^2 + 4\Omega_{\mathrm{m},0}(x^{-3} - 1)}\right]$	H_0, r_0H_0

2 Structural stability issues

The idea of structural stability, emerged in 1930's due to famous Russian scientists Andronov, Leontovich and Pontriagin (called by them "rough systems") [17], who observed that actual state of the system can never be specified exactly and application of dynamical systems might be useful anyway if it can describe features of the phase portrait that persist when state of the system is allowed to move around. Among all dynamicists there is a shared prejudice that

- there is a class of phase portraits that are far simpler than arbitrary ones which can explain why a considerable portion of the mathematical physics has been dominated by the search for the generic properties. The exceptional cases should not arise very often in application and they de facto interrupt discussion;
- physically realistic models of the world should possess some kind of structural stability because to have many dramatically different models all agreeing with observations would be fatal for the empirical methods of science [18].

In cosmology a property (for example acceleration) is believed to be physically realistic if it can be attributed to generic subset of the models within a space of all admissible solutions or if it possesses a certain stability, i.e. if it is shared by a "epsilon perturbed model". The problem is how to define

- a space of states and their equivalence,
- a perturbation of the system.

The dynamical system is called structurally stable if all its δ-perturbations (sufficiently small) have an epsilon equivalent phase portrait. Therefore for the conception of structural stability we consider a δ perturbation of the vector field determined by right hand sides of the system which is small (measured by delta). We also need a conception of epsilon equivalence. This has the form of topological equivalence – a homeomorphism of the state space preserving the arrow of time on each trajectory. In the definition

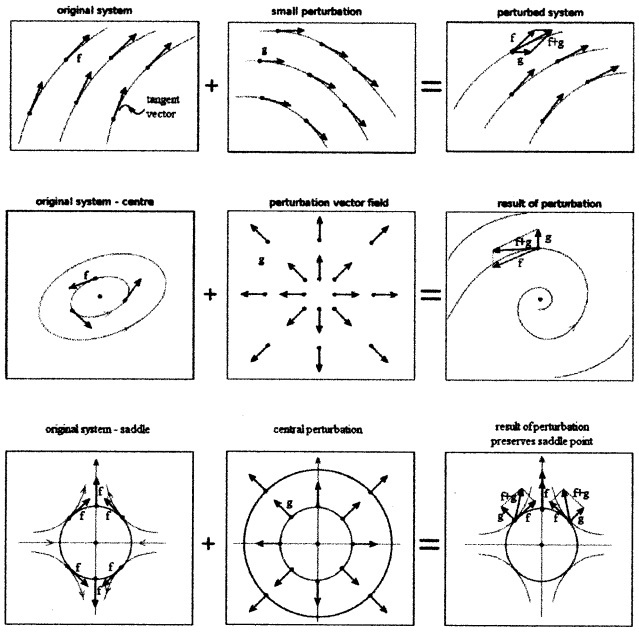

Figure 1: The perturbation of the original vector field f by a small vector g (upper). The effect of perturbation for a centre – the illustration of structural instability (middle). The effect of perturbation of a saddle – the illustration of structural stability (lower).

of structural stability consider only deformation of the "rubber sheet" type stretches or slides of the phase space by a small amount measured by epsilon.

Fig. 1 illustrates the property of structural stability of a single spiral attractor (focus) and saddle point, and the structural instability of center. The addition of a delta perturbation pointing outward (no matter how week) results in a point repellor. We call such a system structurally unstable because the phase portrait of the center and focus are not topologically equivalent (note that all phase curves around the center are closed in contrast to the focus). Hence one can claim that the pendulum system (without friction) is structurally unstable. The idea of structural stability defines the notion of stability of differential deterministic models of physical processes.

Theorem 2.1 (Peixoto) *Structurally stable dynamical systems on the plane form open and dense subsets in the space of all dynamical systems defined on the compact manifold.*

Remark 2.1 *There is no counterpart of Peixoto's theorem in higher dimensions.*

This theorem is a basic characterization of the structurally stable dynamical systems on the plane which offers the possibility of an exact definition of generic (typical) and non-generic (exceptional) cases (properties) employing the notion of structural stability. For our aims it is important that Peixoto's theorem can characterize generic cosmological models in terms of the potential function.

It is easy to test whether a planar polynomial system has a structurally stable global phase portrait. In particular, a vector field on the Poincaré sphere will be structurally unstable if there is a non-hyperbolic critical point at infinity or if there is a trajectory connecting saddles on the equator of the Poincaré sphere S^2. In opposite case if additionally the number of critical points and limit cycles is finite, f is structurally stable on S^2. Following Peixoto's theorem the structural stability is a generic property of the C^1 vector fields on a compact two-dimensional differentiable manifold $\mathcal{M}$.

Definition 2.1 *If the set of all vector fields $f \in C^r(\mathcal{M})$ $(r \geq 1)$ having a certain property contains an open dense subset of $C^r(\mathcal{M})$, then the property is called generic.*

From the physical point of view it is interesting to known whether a certain subset V of $C^r(\mathcal{M})$ (representing the class of cosmological accelerating models in our case) contains a dense subset because it means that this property (acceleration) is typical in V.

Let establish a simple relation between the geometry of the potential function and localization of the critical points and its character for the case of dynamical systems of the Newtonian type:

- The critical points of the systems under consideration $\dot{x} = y$, $y = \frac{\partial V}{\partial x}$ lie always on the x axis, i.e. they represent static universes $y_0 = 0$, $x = x_0$.

- The point $(x_0,\ 0)$ is a critical point of the Newtonian system if it is a critical point of the potential function $V(x)$, i.e. $V(x) = E$ (E is the total energy of the system $E = \frac{y^2}{2} + V(x)$; $E = 0$ for the case of flat models and $E = -\frac{k}{2}$ in general).

- If $(x_0,\ 0)$ is a strict local maximum of $V(x)$, it is a saddle.

- If $(x_0,\ 0)$ is a strict local minimum of the analytic function $V(x)$, it is a center.

- If $(x_0,\ 0)$ is a horizontal inflection point of the $V(x)$, it is a cusp.

- The phase portraits of the Newtonian type systems have reflectional symmetry with respect to the y axis, i.e. $x \to x,\ y \to -y$.

All these properties are simple consequences of the Hartman-Grobman theorem which states that near the non-degenerate (hyperbolic) critical points the original dynamical system is equivalent to its linear part. Therefore the character of a critical point is determined by the eigenvalues of the linearization matrix given by a simple equation $\lambda^2 + \det A = 0$, where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2 V}{\partial x^2} & 0 \end{pmatrix}_{\{x_0,\ 0\}} \tag{2.10}$$

is the linearization matrix.

Hence, in the case of a maximum we obtain a saddle with λ_1, λ_2 real of opposite signs, and if the potential function assumes a minimum at the critical point we have a center with $\lambda_{1,\ 2}$ purely imaginary of mutually conjugate. Therefore, among all distinguished cases, only if the potential function admits a local maximum at the critical point we have a structurally stable global phase portrait. Because $V \leq 0$ and $\partial V/\partial a = \frac{1}{6}(\rho + 3p)(a)$ the Universe is decelerating if the strong energy condition is satisfied and accelerating if the strong energy condition is violated. Hence, among all simple scenarios, the one in which deceleration is followed by acceleration is the only structurally stable one.

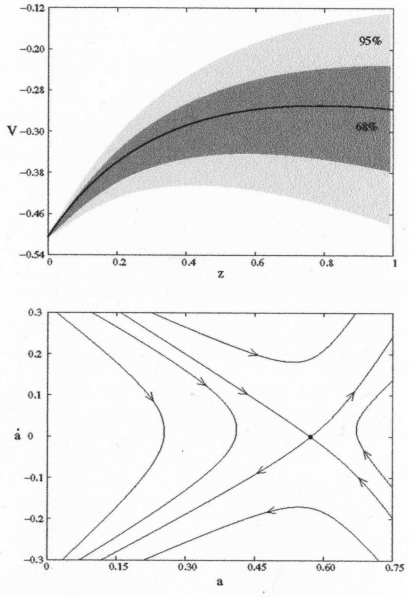

Figure 2: Potential and phase portrait from SNIa data

Due to the existence of a simple relation between the luminosity distance $d_L(z)$ as a function of redshift z and the Hubble function

$$H(z) = \left[\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-1}, \tag{2.11}$$

it is possible to reconstruct the potential function $V(a)$ [19, 20]. Note that even if the luminosity distance is obtained accurately, the potential cannot be determined uniquely as it depends on $\Omega_{k,0}$ – an additional curvature parameter. If we assume that the model is flat, then the trajectories of the corresponding Hamiltonian system lies on the zero energy level and

$$V(z) = -\frac{1}{2}\left[(1+z)\frac{d}{dz}\left(\frac{d_L(z)}{1+z}\right)\right]^{-2}. \tag{2.12}$$

From Fig. 2 we obtain the shape of the potential function like the one of the ΛCDM model. Hence the corresponding phase portrait contains a single

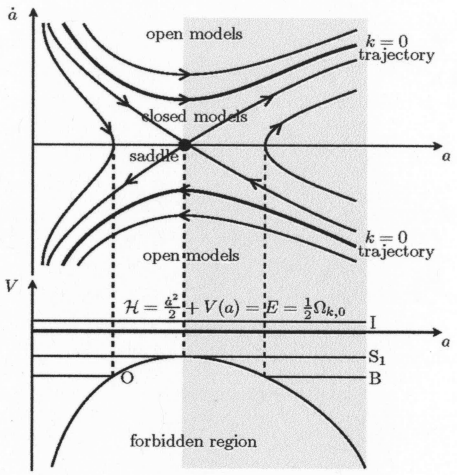

Figure 3: Phase portrait of ΛCDM model

maximum [21]. The value of redshift at this point we call the transition redshift and denote z_T. It is possible to obtain not only the qualitative shape of the potential function (an inverted single-well potential) but also some quantitative attributes of the model. Especially it is well to know when the switch from the deceleration to the acceleration epoch occurs in different cosmological scenarios.

3 Generic and non-generic global phase portrait

The analysis of full dynamical behavior of trajectories requires the study of the behavior of trajectories at infinity. It can be performed by means of the Poincaré sphere construction. In this approach we project the trajectories from center of the unit sphere $S^2 = (X, Y, Z) \in R^3 \colon X^2 + Y^2 + Z^2 = 1$ onto the $(x,\ y)$ plane tangent to S^2 at either the north or south pole. Due to this central projection (introduced by Poincaré) the critical points at infinity are spread out along the equator. Therefore if we project the upper hemisphere

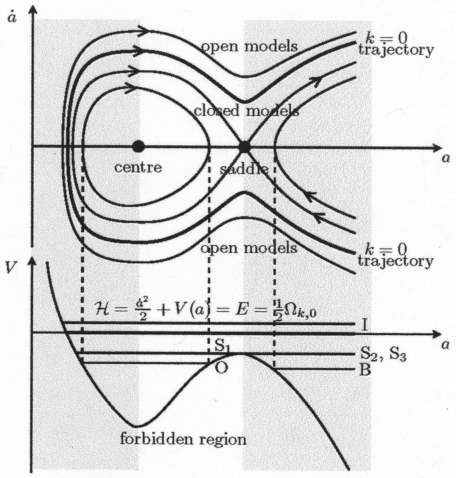

Figure 4: Structural instability of bounce cosmological models

S^2 onto the (x, y) plane of dynamical system of the Newtonian type, then

$$x = \frac{X}{Z}, \quad y = \frac{Y}{Z} \tag{3.13}$$

or

$$X = \frac{x}{\sqrt{1+x^2+\left(\frac{\partial V}{\partial x}\right)^2}}, \quad Y = \frac{y}{\sqrt{1+x^2+\left(\frac{\partial V}{\partial x}\right)^2}}, \quad Z = \frac{z}{\sqrt{1+x^2+\left(\frac{\partial V}{\partial x}\right)^2}}. \tag{3.14}$$

There is a simple way to introduce the metric in the space of all dynamical systems on the compactified plane. If $f \in C^1(\mathcal{M})$ where $\mathcal{M}$ is an open subset of R^n, then the C^1 norm of f can be introduced in a standard way

$$||f||_1 = \sup_{x \in E} |f(x)| + \sup_{x \in E} \|Df(x)\|, \tag{3.15}$$

where $|\ldots|$ and $\|\ldots\|$ denotes the Euclidean norm in R^n and the usual norm of the Jacobi matrix $Df(x)$, respectively. It is well known that the set of vectors field bounded in the C^1 norm forms a Banach space.

It is natural to use the defined norm to measure the distance between any two dynamical systems of the multiverse. If we consider some compact

subset $\mathcal{K}$ of $\mathcal{M}$ then the C^1 norm of vector field f on $\mathcal{K}$ can be defined as

$$\|f\|_1 = \max_{x\in\mathcal{K}} |f(x)| + \max_{x\in\mathcal{K}} \|Df(x)\| < \infty. \tag{3.16}$$

Let $E = R^n$ then the ϵ-perturbation of f is the function $g \in C^1(\mathcal{M})$ form which $\|f-g\| < \epsilon$. The introduced language is suitable to reformulate the idea of structural stability given by Andronov and Pontryagin. The intuition is that f should be structurally stable vector field if for any vector field g near f, the vector fields f and g are topologically equivalent. A vector field $f \in C^1(\mathcal{M})$ is said to be structurally stable if there is an $\epsilon > 0$ such that for all $g \in C^1(\mathcal{M})$ with $\|f-g\|_1 < \epsilon$, f and g are topologically equivalent on open subsets of R^n called $\mathcal{M}$. To show that system is not structurally stable on R^n it is sufficient to show that f is not structurally stable on some compact $\mathcal{K}$ with nonempty interior.

If we consider some subclass of dark energy models described by the vector field $[y, -\partial V/\partial x]^T$ on the Poincaré sphere, then the right hand sides of the corresponding dynamical systems are of the polynomial form of degree m. Then f is structurally stable if

(i) the number of critical points and limit cycles is finite and each critical point is hyperbolic . therefore a saddle point in finite domain,

(ii) there are no trajectories connecting saddle points. It is important that if the polynomial vector field f is structurally stable on the Poincaré sphere S^2 then the corresponding polynomial vector field $[y, -\partial V/\partial x]^T$ is structurally stable on R^2.

Following Peixoto's theorem the structural stability is a generic property of C^1 vector fields on a compact two-dimensional differentiable manifold $\mathcal{M}$. If a vector field $f \in C^1(\mathcal{M})$ is not structurally stable it belongs to the bifurcation set $C^1(\mathcal{M})$. For such systems their global phase portrait changes as vector field passes through a point in the bifurcation set.

Therefore, in the class of dynamical systems on the compact manifold, the structurally stable systems are typical (generic) whereas structurally unstable are rather exceptional. In science modeling, both types of systems are used. While the structurally stable models describe "stable configuration" structurally unstable model can describe a fragile physical situation which require fine tuning.

The Sobolev metric introduced in the multiverse of dark energy models can be used to measure how far different cosmological model with dark energy are to the canonical ΛCDM model. For this aim let us consider a different dark energy models with dust matter and dark energy. We also for simplicity of presentation assume for all models have the same value of $\Omega_{m,0}$ parameters which can be obtained, for example, from independent extragalactic measurements. Then, the distance between any two cosmological model, say model '1' and model '2' is

$$d(1,\ 2) = \max_{x \in C}[|V_{1x} - V_{2x}|,\ |V_{1xx} - V_{2xx}|] \tag{3.17}$$

where we assumed the same value of the parameter H_0 measured at the present epoch for all cosmological models which we compare, V_1 and V_2 and their derivatives are only the parts of the potentials without the matter term.

4 New paradigm in cosmology – matter content in terms of scalar field

An alternative method of describing the matter content of the Universe is to adopt the energy momentum tensor of a scalar field ϕ with the kinetic part and the potential $V = V(\phi)$. The energy momentum tensor is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}[(\partial\phi)^2 - V(\phi)]. \tag{4.18}$$

If the FRW metric is postulated (cosmological principle) then the scalar field are homogeneous, i.e. $\phi = \phi(t)$. Formally the cosmology with the homogeneous scalar fields can be treated as a cosmology with perfect fluid; a source of gravity and energy density and pressure are

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi),\ p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{4.19}$$

Note that if the scalar field is slowly changing in time then the equation of state for the cosmological constant can be reproduced. However in general

the barotropic form of the equation of state $p = p(\rho)$ for the scalar field does not always exist in this special form.

There are many reasons to replace the "paradigm" of describing the matter content in terms of the barotropic equation of state by the description in terms of scalar field. Let us mention some of them.

- The standard cosmological model requires the extension to the Planck epoch where quantum cosmology is dedicated to the description of high energetic states of the Universe evolution. In this description barotropic equation of state seems to be unnatural and preferred are just scalar field instead of barotropic matter.

- In standard cosmology as well (as probably in contemporary particle physics) appeared the problem of vacuum energy. If in cosmology we explain an accelerating phase of expansion of the Universe through the cosmological constant (which we interpret as vacuum energy) then theoretical expectations differs from the supernovae observation by 50-100 orders of magnitude. We are looking for some physical mechanism of vacuum decaying going toward the small value which will in order drive acceleration of the current Universe.

The idea of quintessence is offered in this context as the solution of the cosmological constant problem, i.e. an answer for the question why the value of the cosmological constant (vacuum energy) is so small in the current Universe. Instead of $\Lambda(t)$, which is changing in time and becomes inconsistent with the energy conservation condition, we postulate the presence of a scalar field with the potential $V(\phi)$, minimally or non-minimally coupled to gravity. In order to satisfy the conservation energy momentum tensor a massless scalar field is necessary. The condition of conservation energy-momentum tensor for the non-minimally scalar field in the FRW universe reduces to equation of motion (the Klein-Gordon equation)

$$\ddot{\phi} + 3H\dot{\phi} + U'(\phi, R) = 0, \tag{4.20}$$

where the term of non-minimal coupling to gravity was taken in the very simple form; $U = \xi R\frac{\phi^2}{2}$; $' \equiv \frac{\partial}{\partial\phi}$ denotes partial differentiation with respect the scalar field; R denotes the Ricci scalar.

Formally the effects of the non-minimal coupling constant ξ can be treated also as an effect of perfect fluid with the energy density and pressure in the form

$$\rho_\phi = \epsilon\frac{\dot\phi^2}{2} + V(\phi) + 3\epsilon\xi H^2\phi^2 + 3\epsilon\xi H(\phi^2)^{\cdot}, \tag{4.21}$$

$$p_\phi = \epsilon(1-4\xi)\frac{\dot\phi^2}{2} - V(\phi) + \epsilon\xi H(\phi^2)^{\cdot} - 2\epsilon\xi(1-6\xi)\dot H\phi^2 - 3\epsilon\xi(1-8\xi)H^2\phi^2, \tag{4.22}$$

where $\epsilon = \pm 1$ for canonical and phantom scalar fields, respectively.

In the general if we assume the matter content in terms of the scalar field this means the exit beyond the standard cosmological model. In some our future investigation we will assume the presence of both kind of matter content described by the barotropic form of the equation of state and the scalar field. Then one can discover new interesting effects which are not present in the cosmological evolution if scalar field as well as matter are treated separately.

In this scalar field "paradigm" some problems, related with undetermination of the potential of the scalar field, appear in the description of matter content. Which potential for the scalar field should be taken into account? Will the evolutional scenarios be sensitive on the choice of a form the potential? These questions are very important for consistency of SFC.

While some forms of potential (for example quadratic, exponential, etc.) are very popular in SFC there is not theoretical or observational reasons indicating which uniquely should determine our choice? With this before addressed problem connected is another one: how treat or interpret the Klein-Gordon equation? Are they equation motion for scalar field or they are just dedicated for determination of potential function?

Main advantage of representing the cosmic evolution in terms of a dynamical system is the possibility of its investigation for all admissible initial conditions. It is important especially in cosmology because initial conditions for the Universe are unknown. In some sense cosmology is a physical theory of the Universe as well as a theory of initial conditions which lead to the current Universe.

Moreover the dynamical system methods allow us to analyze the stability of evolutional paths in the phase space.

In the scalar field cosmology with the non-minimal coupling to gravity of type $\frac{1}{2}\xi R\phi^2$ the action of gravity and matter assumes the following form

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(\frac{1}{R} - \epsilon(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R\phi^2) - 2V(\phi)\right) + S_m. \quad (4.23)$$

The equation of motion for the scalar field are obtained after variation of action under the scalar field and then we obtain

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \epsilon V'(\phi) = 0, \quad (4.24)$$

where ξ is the non-minimal coupling constant, R is the Ricci scalar and $\epsilon = \pm 1$ for the canonical and phantom scalar fields, a prime denotes differentiation with respect to ϕ, $' \equiv \frac{d}{d\phi}$.

Taking variation under the FRW metric we obtain the FRW equation with the first integral in the form

$$E = \epsilon\frac{1}{2}\dot{\phi}^2 + \epsilon 3\xi H^2\phi^2 + \epsilon 3\xi H(\phi^2)^{\cdot} + V(\phi) + \rho_m - 3H^2 \quad (4.25)$$

or

$$3H^2 = \rho_\phi + \rho_m \quad (4.26)$$

for a flat universe.

The acceleration equation has the following form

$$\dot{H} = -\frac{1}{2}[(\rho_\phi + p_\phi) + \rho_m(1 + w_m)], \quad (4.27)$$

where

$$\rho_\phi = \epsilon\frac{1}{2}\dot{\phi}^2 + V(\phi) + \epsilon 3\xi H^2\phi^2 + \epsilon 3\xi H(\phi^2)^{\cdot}, \quad (4.28)$$

$$p_\phi = \epsilon\frac{1}{2}(1 - 4\xi)\dot{\phi}^2 - V(\phi) + \epsilon\xi H(\phi^2)^{\cdot} - \epsilon 2\xi(1 - 6\xi)\dot{H}\phi^2 - \\ - \epsilon 3\xi(1 - 8\xi)H^2\phi^2 + 2\xi\phi V'(\phi). \quad (4.29)$$

w_m is a coefficient in the equation of state $p_m = w_m\rho$ for matter. It would be convenient to introduce new state variables

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H},\ y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}H},\ z \equiv \frac{\kappa}{\sqrt{6}}\phi \tag{4.30}$$

for representing of the state of the system in the phase space.

The acceleration equation can be rewritten to a new form in which is present fluid with the effective energy density ρ_{eff} and effective pressure p_{eff}

$$\dot{H} = \frac{\kappa^2}{2}(\rho_{\text{eff}} + p_{\text{eff}}) = -\frac{3}{2}H^2(1 + w_{\text{eff}}), \tag{4.31}$$

where

$$w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{1}{1 - \epsilon 6\xi(1 - 6\xi)z^2}$$
$$[-1 + \epsilon(1 - 6\xi)(1 - w_m)x^2 + \epsilon 2\xi(1 - 3w_m)(x + z)^2 + (1 + w_m)(1 - y^2) -$$
$$- \epsilon 2\xi(1 - 6\xi)z^2 - 2\xi\lambda y^2 z], \tag{4.32}$$

where $\lambda = -\sqrt{6}\frac{1}{V(\phi)}\frac{dV(\phi)}{d\phi}$ is a quantity which characterizes geometry of the potential.

The dynamics of the model under consideration can be represented as the following 4-dimensional dynamical system

$$x' = -(x - \epsilon\frac{1}{2}\lambda y^2)$$
$$[1 - \epsilon 6\xi(1 - 6\xi)z^2] + \frac{3}{2}(x + 6\xi z)[-\frac{4}{3} - 2\xi\lambda y^2 z +$$
$$+ \epsilon(1 - 6\xi)(1 - w_m)x^2 + \epsilon 2\xi(1 - 3w_m)(x + z)^2 + (1 + w_m)(1 - y^2)]), \tag{4.33}$$

$$y' = y(2 - \frac{1}{2}\lambda x)$$
$$[1 - \epsilon 6\xi(1 - 6\xi)z^2] + \frac{3}{2}y[-\frac{4}{3} - 2\xi y^2 z +$$
$$+ \epsilon(1 - 6\xi)(1 - w_m)x^2 + \epsilon 2\xi(1 - 3w_m)(x + z)^2 + (1 + w_m)(1 - y^2)], \tag{4.34}$$

$$z' = x[1 - \epsilon 6\xi(1 - 6\xi)z^2], \tag{4.35}$$

$$\lambda' = -\lambda^2(\Gamma - 1)x[1 - \epsilon 6\xi(1 - 6\xi)z^2], \tag{4.36}$$

where $\Gamma = \frac{\frac{d^2V(\phi)}{d\phi^2}V(\phi)}{(\frac{dV(\phi)}{d\phi})^2}$ and the original time variable is replaced by a new time parameter τ

$$\frac{d}{d\tau} = [1 - \epsilon 6\xi(1 - 6\xi)z^2]\frac{d}{d\ln a}. \tag{4.37}$$

In Fig. 5 trajectories represent a twister type solution interpolating between the radiation dominated universe, the matter dominated universe and accelerating universe [22].

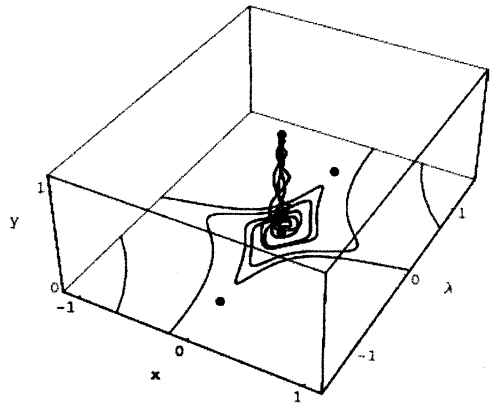

Figure 5: Interpolating cosmological epochs – a new twister solution.

Definition 4.1 (Ensemble of SFC models) *By ensemble $\mathcal{E}$ of homogeneous and isotropic cosmological models (i.e. FRW models) we understand*

the space of all dynamical systems $\dot{x} = f(x)$, where $x \in R^n$ is state variable and dot denotes differentiation with respect time t (or its monotonic function: $t \to \tau$: $dt = \Psi(x)d\tau$; ψ is C^∞ class). A vector field f is C^1 vector field defined on the open subset E of R^n; a positional variable x which satisfies the Einstein field equation with the source of gravity in the form of matter and scalar field, minimally and non-minimally coupled to gravity.

Definition 4.2 (Equivalence of ensembles) *We say that two ensembles $\mathcal{E}$ and $\mathcal{E}'$ are equivalent if there is the homeomorphism h transforming the vector field, say $f(x)$ into $g(x)$ (of the same dimension) with the orientation preservation. Of course, if we have find a diffeomorphism sending integral curves of $f(x)$ into those of $g(x)$ then corresponding ensembles we treated as equivalent.*

Therefore one can introduce equivalence classes and then the space of all dynamical system in R^n is divided on disjoint classes of abstraction (equivalence classes).

In the future if we think about elements of ensemble we makes implicate assumption that we are dealing only with the represent of the equivalence relation. In the cosmological applications the vector field becomes a function of state variables and can also depends on the model parameters.

5 Structural stability of SFC

In the dynamical system theory our investigations concentrated rather on studies properties of global portraits than on investigations of individual solutions (phase space trajectories). In this context notion of ensemble $\mathcal{E}$ appears as natural way for study how typical generic are cosmological models dedicating for solution of different cosmological problems.

Because our models are only an effective description of the reality we believe that they are generic in ensemble. We understand that they are only approximation and therefore they are realistic if they can be attributed not only to individual (non-generic) subsets of ensemble.

Our theoretical presumption of methodological nature is that all models are rather generic than fine tunning and therefore non-typical in an ensemble.

In this case there is simple way to introducing the metric d in the ensemble $\mathcal{E}$ of all dynamical systems in the plane:

$$\begin{aligned} d(U_1,U_2) = \max\{&\sup_{x,y\in U_i}|f^1(x,y)-g^1(x,y)|,\\ &\sup_{x,y\in U_i}|\tfrac{\partial f^1}{\partial x}-\tfrac{\partial g^1}{\partial x}|,\\ &\sup_{x,y\in U_i}|\tfrac{\partial f^1}{\partial y}-\tfrac{\partial f^1}{\partial y}|,\\ &\sup_{x,y\in U_i}|f^2(x,y)-g^2(x,y)|,\\ &sup_{x,y\in U_i}|\tfrac{\partial f^2}{\partial x}-\tfrac{\partial f^2}{\partial x}|,\\ &\sup_{x,y\in U_i}|f^2(x,y)-g^2(x,y)|,\\ &sup_{x,y\in U_i}|\tfrac{\partial f^2}{\partial y}-\tfrac{\partial g^2}{\partial y}|\}, \end{aligned} \tag{5.38}$$

where $U_1,...,U_k$ open sets be a covering of $M \in R^2$ [20]; dynamical systems have the form $U_1\colon \frac{dx}{dt} = f^1(x,y), \frac{dy}{dt} = f^2(x,y)$.

Definition 5.1 (Closeness of planar dynamical system) *By closeness of dynamical systems we mean closeness along with the first derivatives for two systems*

$$U_1\colon \frac{dx}{dt} = f^1(x,y), \frac{dy}{dt} = f^2(x,y), \tag{5.39}$$

$$U_2\colon \frac{dx}{dt} = g^1(x,y), \frac{dy}{dt} = g^2(x,y). \tag{5.40}$$

Definition 5.2 (Structural stability) *Therefore system of differential equations is structurally stable if every sufficiently close system has the some phase portrait.*

This construction gives us a simple method of investigation of behavior of the system at infinity. The phase space is compact by adding the circle at infinity, i.e. it is the projective plane.

Theorem 5.1 (Andronov-Pontriagin) *Planar systems are structurally stable if and only if:*

- *all its critical points are hyperbolic (i.e. stable, unstable or saddles),*
- *all periodic orbits have a multiplier are different from 1,*

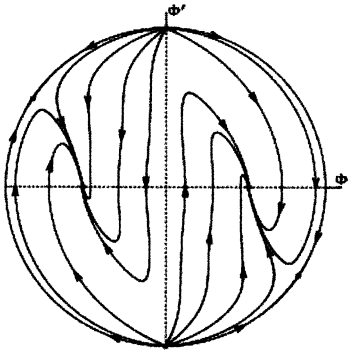

Figure 6: The phase portrait for the phantom scalar field for coupling constant $\frac{3}{25} < \xi < \frac{1}{6}$. The system is structurally unstable because the presence of saddle-node critical point at the circle at infinity.

- *there is not phase curves which connect saddles.*

In the special case of subclass of cosmological models which evolution's described by some vector field $[f^1, f^2]^T$ on the Poincaré sphere, where f^1 and f^2 are polynomials of degree m there is simple method of detection structurally stable (and generic) dynamical systems. Vector field f is structurally stable if

- the number of critical points and limit cycles is finite,
- there is no trajectories connecting saddle points.

Note that the presence of non-hyperbolic critical points on the phase portrait is a sufficient condition for structural instability. It can be also shown that if the polynomial vector field is structurally stable on the Poincaré sphere S^2, then the corresponding vector field $f = [f^1, f^2]^T$ is structurally stable on R^2.

Let us discuss the phase portraits of flat cosmological systems with the scalar field and matter in details. The phase variables are so called energetic variables. We consider different cases of the models with the canonical and phantom scalar field with a quadratic potential.

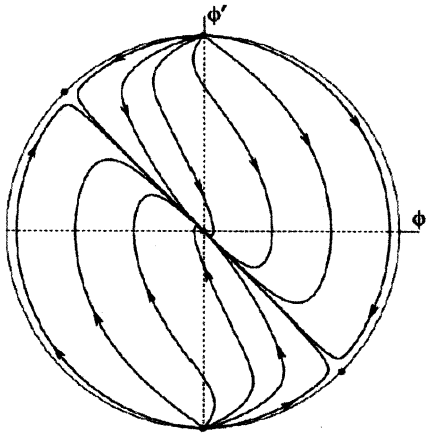

Figure 7: The phase portrait for $\epsilon = -1$, $\xi = \frac{1}{6}$.

From the point of view of structural stability the ΛCDM model and its copy in scalar field cosmology becomes structurally stable. We are looking for SFC models with such potential form which makes the system structurally stable.

We observe that phase portraits changes dramatically under changing the form of potential. They are sensitive depending on the choice of potential.

The characteristic is the following scenario us of bifurcation saddle-node point:

- saddle-node critical point located at the circle at infinity bifurcates to saddle and node separately.
- critical point under small perturbation vanishes and does not appear on the phase portrait.

The above scenario illustrates how the SFC models make structurally stable systems by introduction some perturbation in the form of the non-minimal coupling constant. Different phase portraits illustrated road toward structurally stable models via a non-minimal coupling constant effect.

6 Conclusion

- Description of matter content in terms of both barotropic matter and scalar field gives rise to the new theoretically interesting solutions like the twister solution for example.
- The standard cosmological model is structurally stable and this fact can explain why it is still the best description of the Universe at the accelerating phase of its expansion.
- We are looking for a cosmological model filled by both matter and scalar field which are structurally stable. This program can be fully realized for cosmological dynamical system on the plane.
- Issue of structural stability and ensemble of cosmological models can be useful in searching for the potential form for the cosmological scalar field.

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METHOD OF QUARK BAG MODEL AND ITS COSMOLOGICAL APPLICATIONS

Maxim Brilenkov
Department of Theoretical Physics
Odessa National University
St. Dvoryanskaya 2, 65082 Odessa, Ukraine
maxim.brilenkov@gmail.com
Maxim Eigorn
CREST and NASA Research Centers
North Carolina Central University,
Duhram, NC, U.S.A.
maxim.eigorn@gmail.com
and
Atronomical Observatory
Odessa National University
St. Dvoryanskaya 2, 65082 Odessa, Ukraine
maxim.eigorn@gmail.com
Alexander Zhuk,
Atronomical Observatory
Odessa National University
St. Dvoryanskaya 2, 65082 Odessa, Ukraine
ai.zhuk2@gmail.com,
Laszlo Jenkovszky
BITP, Academy of Science of Ukraine
03680 Kiev, Ukraine
and
Wigner Research Centre for Physics
Budapest, Hungary
jenk@bitp.kiev.ua

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Abstract

We consider the cosmological applications of the quark-gluon plasma branch of bag models. The nonlinear nature of the corresponding equation of state offers various interesting possibilities to introduce a perfect fluid as dark energy rather that the cosmological constant. We discuss this possibility, finding the corresponding solutions of the Friedmann equations and comparing their predictions with the experimental data of cosmological observations (in particular, the present values of the Hubble and deceleration parameters).

Keywords: quark gluon plasma, dark energy

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1 Introduction

Over two decades ago [1, 2] (see also [3]) the accelerated expansion of the early Universe was derived from a quark bag model with the proper equations of state (EoS). It was called tepid [1, 2] or little [3] inflation, in view of its moderate scales, compared to the better known earlier inflation. However, occurring at a later time (when the temperature $\sim$ 200 MeV) and smearing a lot of the earlier effects, it may have important consequences for the observable Universe.

The derivation was based on a quark-gluon bag EoS completing the Friedmann equations. Our Universe was cooling down along the "hot" (i.e. quark-gluon) branch of the EoS until it reached the point/area of the transition to the confined hadron phase. In most of the papers along these lines (see [1] and references therein) the process of inflation terminates by a phase transition (hadronization) to the state of colorless objects. The details of the phase transition, depending on unknown confining forces, are poorly known and leave much room for speculations.

In this paper we consider the possibility that a very small fraction of colored objects – quarks and gluons – survived after the phase transition in the form of a perfect fluid uniformly spread within the colorless hadronic medium. We suppose that this fluid has the same thermodynamical properties as the quark-gluon plasma (QGP). Therefore, we call it as a QGP-like perfect fluid. It is of interest to investigate cosmological consequences of such assumption. In our paper, we demonstrate that such fluid can provide an alternative (with respect to the cosmological constant) explanation to the late-time accelerating expansion of the Universe. It is worth mentioning that quark-gluon plasma as dark matter in the halo of galaxies was investigated in the paper [4]. The authors arrived at a very interesting conclusion that flatness of the rotational curves can be explained due to the presence of quark matter in halos (because of additional attraction from such matter). In our paper, we spread this quark matter over the whole Universe and demonstrate that it can result in the late time acceleration of the Universe. Analogously, the authors of [4] also claimed that quark-gluon plasma on the global level behaves like dark energy.

The paper is structured as follows. In Sec. 2, we consider the influence

of QGP-like perfect fluid on the late-time expansion of the Universe. The main results are briefly summarized in concluding Sec. 3.

2 QGP-like perfect fluid

We suppose a model where a part of quark gluon plasma survived after the phase transition in the form of the homogeneously and isotropically distributed perfect fluid. We do not know what part of QGP survived (see however some estimate at the very end of this section). Maybe, it is more correct to speak about some unknown perfect fluid with the thermodynamical functions motivated by the QGP. Therefore, we call this fluid as a QGP-like perfect fluid. We are going to investigate the cosmological consequences of such proposal.

In general, we can consider the pressure of the form

$$P(T) = \widehat{A}_0 + \widehat{A}_1 T + \widehat{A}_2 T^2 + \widehat{A}_3 T^3 + \widehat{A}_4 T^4 , \tag{2.1}$$

For vanishing chemical potential, $\mu = 0$, one can easily calculate the remaining thermodynamical quantities, e.g., for the energy density we have

$$\mathcal{E}(T) = T\frac{dP}{dT} - P , \tag{2.2}$$

so, as the result

$$\mathcal{E}(T) = -\widehat{A}_0 + \widehat{A}_2 T^2 + 2\widehat{A}_3 T^3 + 3\widehat{A}_4 T^4 . \tag{2.3}$$

The energy conservation equation

$$d(\mathcal{E}a^3) + Pd(a^3) = 0 \tag{2.4}$$

leads to the differential equation

$$\frac{da}{a} = -\frac{\left(2\widehat{A}_2 + 6\widehat{A}_3 T + 12\widehat{A}_4 T^2\right)}{3\left(\widehat{A}_1 + 2\widehat{A}_2 T + 3\widehat{A}_3 T^2 + 4\widehat{A}_4 T^3\right)} dT . \tag{2.5}$$

The solution of this equation enables to determine the dependence $T = T(a)$. Unfortunately, there is no solution of (2.5) in elementary functions. Therefore, we consider two particular models.

2.1 Model I

First, we consider the case $\widehat{A}_0 = 0$, $\widehat{A}_2 = 0$, $\widehat{A}_3 = 0$. The pressure is the same in any point of QGP, and the coefficients $\widehat{A}_1$ and $\widehat{A}_4$ satisfy the following relation:

$$\frac{\widehat{A}_1}{\widehat{A}_4} = -0.8114\, T_c^3 . \tag{2.6}$$

Therefore, we consider the case $\widehat{A}_1 < 0$ and $\widehat{A}_4 > 0$. The coefficients $\widehat{A}_1$ and $\widehat{A}_4$ have dimensions cm^{-3} and $\mathrm{erg}^{-3}\mathrm{cm}^{-3}$, respectively.

Integrating (2.5), we get

$$T = \left(\frac{\widehat{C}^3 - \widehat{A}_1 a^3}{4\widehat{A}_4 a^3}\right)^{1/3} \quad \Rightarrow \quad \varepsilon(T) = 3\widehat{A}_4 \left(\frac{\widehat{C}^3 - \widehat{A}_1 a^3}{4\widehat{A}_4 a^3}\right)^{4/3} , \tag{2.7}$$

where $\widehat{C} \geq 0$ is the dimensionless constant of integration which is defined by the temperature T_0 and the scale factor a_0 at the present time:

$$\widehat{C} = \left(\widehat{A}_1 + 4\widehat{A}_4 T_0^3\right)^{1/3} a_0 = \widehat{A}_4^{1/3} \left(-0.8114\, T_c^3 + 4T_0^3\right)^{1/3} a_0 . \tag{2.8}$$

The temperature T tends to the constant value when the scale factor approaches infinity

$$T \longrightarrow T_\infty = \left(\frac{-\widehat{A}_1}{4\widehat{A}_4}\right)^{1/3} = 0.5876\, T_c \quad \text{for} \quad a \to \infty . \tag{2.9}$$

It can be easily verified that, in the limit $a \to \infty$, the energy density $\varepsilon \to (3/4)[\widehat{A}_1^4/(4\widehat{A}_4)]^{1/3}$ and the pressure $p \to -(3/4)[\widehat{A}_1^4/(4\widehat{A}_4)]^{1/3}$, i.e. the perfect fluid has asymptotically the vacuum-like equation of state $p = -\varepsilon$. If $\widehat{C} \equiv 0$, then the perfect fluid has this equation of state for all time.

Let us consider the first Friedmann equation in the form

$$3\frac{\mathcal{H}^2 + K}{a^2} = \kappa\mathcal{E} + \kappa\varepsilon_0^{\mathrm{mat}} \left(\frac{a_0}{a}\right)^3 + \Lambda , \tag{2.10}$$

where we take into account the cosmological constant Λ and the (usual + dark) matter with the present value of the energy density $\varepsilon_0^{\mathrm{mat}}$. In (2.10),

$\mathcal{H} = a'/a = (da/d\eta)/a$, $\kappa = 8\pi G_N/c^4$, G_N is the gravitational constant and $K = \pm 1, 0$ is the spatial curvature. The conformal time η is connected with the synchronous time t: $ad\eta = cdt$. Taking into account Eqs. (2.1) and (2.3), we get for the Hubble parameter $H = (1/a)da/dt = (c/a^2)da/d\eta$ the following expression:

$$H^2 = H_0^2 \left\{ \left[\beta \left(\frac{a_0}{a} \right)^3 + \gamma \right]^{4/3} + \Omega_M \left(\frac{a_0}{a} \right)^3 + \Omega_\Lambda + \Omega_K \left(\frac{a_0}{a} \right)^2 \right\}, \quad (2.11)$$

where

$$\beta = \frac{\widehat{C}^3}{4(a_0^3 \widehat{A}_4)^{1/4}} \left(\frac{\kappa c^2}{a_0^3 H_0^2} \right)^{3/4}, \quad \gamma = -\frac{\widehat{A}_1 a_0^3}{4(a_0^3 \widehat{A}_4)^{1/4}} \left(\frac{\kappa c^2}{a_0^3 H_0^2} \right)^{3/4}. \quad (2.12)$$

We use Eq. (2.11) to find the age of the Universe:

$$-\widetilde{t}_0 = \int_1^0 \frac{\widetilde{a} d\widetilde{a}}{\sqrt{(\beta + \gamma \widetilde{a}^3)^{4/3} + \Omega_M \widetilde{a} + \Omega_\Lambda \widetilde{a}^4 + \Omega_K \widetilde{a}^2}}. \quad (2.13)$$

Considering the second Friedmann equation in the form

$$\frac{2\mathcal{H}' + \mathcal{H}^2 + K}{a^2} = -\kappa P + \Lambda, \quad (2.14)$$

after some obvious algebra we obtain the deceleration parameter

$$\begin{aligned} -q &= \left(\frac{H_0}{H} \right)^2 \left\{ 2\gamma \left[\beta \left(\frac{a_0}{a} \right)^3 + \gamma \right]^{1/3} \right. \\ &- \left. \left[\beta \left(\frac{a_0}{a} \right)^3 + \gamma \right]^{4/3} - \frac{1}{2} \Omega_M \left(\frac{a_0}{a} \right)^3 + \Omega_\Lambda \right\}. \quad (2.15) \end{aligned}$$

At the present time t_0, Eqs. (2.11) and (2.15) read

$$1 = (\beta + \gamma)^{4/3} + \Omega_M + \Omega_\Lambda + \Omega_K, \quad (2.16)$$

$$-q_0 = 2\gamma (\beta + \gamma)^{1/3} - (\beta + \gamma)^{4/3} - \frac{1}{2} \Omega_M + \Omega_\Lambda. \quad (2.17)$$

The latter equation indicates that the acceleration of the Universe (i.e. $-q_0 > 0$) in this model can originate from the first and the last terms on the right-hand side of this equation. Up to now, the nature of the cosmological constant is still unclear. So, we try to explain the acceleration without it, i.e. we suppose that $\Omega_\Lambda = 0$. It is also well known that our Universe is very flat [5, 6]. Hence, we put $\Omega_K = 0$.

Let us consider two particular cases. The first one corresponds to the choice $\beta = 0$. In this case, Eqs. (2.16) and (2.17) take the form

$$1 = \Omega_M + \Omega_{\Lambda,qgp}\,, \tag{2.18}$$

$$-q_0 = -\frac{1}{2}\Omega_M + \Omega_{\Lambda,qgp}\,, \tag{2.19}$$

where $\Omega_{\Lambda,qgp} \equiv \gamma^{4/3}$. These equations reproduce exactly the standard ΛCDM model. Therefore, if we take $-q_0 \approx 0.595$ and $\Omega_M \approx 0.27$, then we get $\Omega_{\Lambda,qgp} \approx 0.73$ and for the age of the Universe $\widetilde{t_0} \approx 1 \Rightarrow t_0 \approx H_0^{-1} \sim 13.7 \times 10^9$yr.

The second case with $\beta \neq 0$ is a bit more complicated, but also more interesting. Because the nature of dark matter is unclear and, according to the observations [5, 6], the visible matter has $\Omega_M \approx 0.04$, we shall take this value as a total contribution of matter. The other experimental restriction follows from the age of globular clusters which is $11 \div 16$ Gyr [7]. Therefore, the Universe cannot be younger. Now, we shall demonstrate that in our model we can satisfy this limitation if we suppose only $\Omega_M \approx 0.04$ and $-q_0 \approx 0.595$ (as in the ΛCDM model).

From Eqs. (2.16) and (2.17) (where $\Omega_\Lambda = \Omega_K = 0$), we express the parameters β and γ via Ω_M and q_0:

$$\beta = \frac{2 - 3\Omega_M + 2q_0}{4(1-\Omega_M)^{1/4}}\,, \quad \gamma = \frac{2 - \Omega_M - 2q_0}{4(1-\Omega_M)^{1/4}}\,. \tag{2.20}$$

Then, for $\Omega_M \approx 0.04$ and $-q_0 \approx 0.595$, we get $\beta \approx 0.174$ and $\gamma \approx 0.796$. For these values of β and γ, the age of the Universe is $\widetilde{t_0} \approx 0.892 \Rightarrow t \approx 12.2$ Gyr, in agreement with the experimental data. Roughly speaking, the parameter γ is responsible for the accelerated expansion of the Universe, and the parameter β plays the role of dark substance.

2.2 Model II

Let us consider now the case $\widehat{A}_1 = 0$, $\widehat{A}_2 = 0$, $\widehat{A}_3 = 0$. The coefficients $\widehat{A}_0$ and $\widehat{A}_4$ satisfy the condition similar to (2.6):

$$\frac{\widehat{A}_0}{\widehat{A}_4} = -0.8114\, T_c^4, \tag{2.21}$$

where $\widehat{A}_0 < 0$ and $\widehat{A}_4 > 0$, and they have dimensions erg $\times$ cm^{-3} and erg^{-3}cm^{-3}, respectively.

Integrating (2.5), we get

$$T = \frac{\overline{C}}{a}, \tag{2.22}$$

where $\overline{C} \geq 0$ is the constant of integration which is defined by the temperature T_0 and the scale factor a_0 at the present time: $C = a_0 T_0$. Therefore, the pressure and energy density depend on the scale factor as follows:

$$\varepsilon = -\widehat{A}_0 + 3\widehat{A}_4 \left(\frac{\overline{C}}{a}\right)^4, \quad p = \widehat{A}_0 + \widehat{A}_4 \left(\frac{\overline{C}}{a}\right)^4. \tag{2.23}$$

Hence, such perfect fluid can be formally considered as a mixture of vacuum and radiation. This conclusion is also confirmed by the form of the Friedmann equations for this model:

$$H^2 = H_0^2 \left[\beta \left(\frac{a_0}{a}\right)^4 + \gamma + \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda + \Omega_K \left(\frac{a_0}{a}\right)^2\right], \tag{2.24}$$

$$-q = \left(\frac{H_0}{H}\right)^2 \left[\gamma - \beta \left(\frac{a_0}{a}\right)^4 - \frac{1}{2}\Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right], \tag{2.25}$$

where

$$\beta = \frac{\widehat{A}_4 \overline{C}^4}{a_0} \left(\frac{\kappa c^2}{a_0^3 H_0^2}\right), \quad \gamma = -\frac{\widehat{A}_0 a_0^3}{3} \left(\frac{\kappa c^2}{a_0^3 H_0^2}\right). \tag{2.26}$$

For the age of the Universe we have

$$-\widetilde{t}_0 = \int_1^0 \frac{\widetilde{a} d\widetilde{a}}{\sqrt{\beta + \gamma \widetilde{a}^4 + \Omega_M \widetilde{a} + \Omega_\Lambda \widetilde{a}^4 + \Omega_K \widetilde{a}^2}}. \tag{2.27}$$

Obviously, the parameter γ plays the role of the cosmological constant. So, we may omit Ω_Λ in above equations. According to the observations, we may also put $\Omega_K = 0$ because of its smallness. We restore exactly the ΛCDM model, e.g., with the choice $\beta = 0$ (then, $\gamma \approx 0.73$). Therefore, in this model the cosmological constant arises due to QGP. Let us estimate the fraction of QGP which should remain after the phase transition to get the observable acceleration. It is clear that parameters $\bar{A}_0$ and $\widehat{A}_0$ play the role of the vacuum energy density before and after the phase transition, respectively: $\varepsilon_{V,in} = -\bar{A}_0$, $\varepsilon_{V,fin} = -\widehat{A}_0$. The initial vacuum energy density $\varepsilon_{V,in} = -\bar{A}_0 \approx 1.42 T_c^4 \sim 2 \times 10^9 \mathrm{MeV}^4$ [8]. For $\gamma \approx 0.73$ from (2.26) we get $\varepsilon_{V,fin} = -\widehat{A}_0 \sim 6 \times 10^{-9} \mathrm{erg} \times \mathrm{cm}^{-3} \approx 3 \times 10^{-35} \mathrm{MeV}^4$. Thus, $\varepsilon_{V,fin}/\varepsilon_{V,in} \sim 10^{-44}$.

3 Conclusions

Our paper was devoted to the great challenge of modern cosmology and high energy physics dubbed dark energy. Up to now, there is no satisfactory explanation for it. In our paper, we proposed a possible solution to this problem. For this purpose, we considered the expansion of the present Universe, using the "hot", i.e. the quark-gluon branch of the bag EoS. Although we made reference to the role of this type of the EoS during the early universe, namely its inflation phase, here we postponed possible speculations about the continuous evolution of the universe, within the present formalism, from its early, quark-gluon stage to the present days, admitting only the possible continuity in the existence in the present Universe of a small fraction of colored objects – quarks and gluons – which escaped hadronization.

We assumed that a very small fraction of colored objects – quarks and gluons – survived after the phase transition in the form of a perfect fluid, called the QGP-like perfect fluid, uniformly spread within the colorless hadronic medium. Obviously, these cosmological scenarios are defined by EoS of quark gluon plasma (QGP). We focus on two possible Eqs. dubbed Model I and Model II, respectively. We have shown that within considered scenarios, there are no fundamental differences in the obtained conclusions

for these models. We have demonstrated that this fluid can play the role of dark energy providing the late-time accelerating expansion of the Universe for both of the models. Moreover, we defined that, to be in agreement with observations, only 10^{-44} part of the colored objects should survive after the phase transition. Therefore, the considered scenario provides new possible ways of solving the problems of dark energy.

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